THE EQUIVARIANT TAMAGAWA NUMBER CONJECTURE AND THE EXTENDED ABELIAN STARK CONJECTURE

DANIEL VALLIÈRES

ABSTRACT. The goal of this paper is to show that the equivariant Tamagawa number conjecture implies the extended abelian Stark conjecture contained in [12] and [11]. In particular, this gives the first proof of the extended abelian Stark conjecture for the base field $\mathbb{Q}$, since the equivariant Tamagawa number conjecture away from 2 was proved in this context by Burns and Greither in [8] and Flach completed their results at 2 in [13].

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1. INTRODUCTION

In his seminal paper [6], David Burns showed that the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(K)), \mathbb{Z}[G])$ provides a universal approach to other conjectures in number theory concerning the special value $s = 0$ of imprimitive L-functions associated to abelian extensions of global

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fields: the Rubin-Stark conjecture, various conjectures of Gross, and some variations and refinements of those by Aoki, Lee, Tan, by Popescu, and by Tate. His paper is a culmination of ideas contained in [7], [1], [2], [4], and [5]. (Note that in recent papers of various authors, the equivariant Tamagawa number conjecture, abbreviated as the ETNC, is often referred to as the leading term conjecture, abbreviated as the LTC.)

Recently, the setting of the classical abelian rank one Stark conjecture of [24] has been generalized by Erickson and Stark in [12]. The problem of extending this conjecture to include higher order of vanishing situations was tackled by Emmons and Popescu in [11]. Their conjecture constitutes a generalization of the Rubin-Stark conjecture of [23]. The key point here is that the set S involved does not necessarily contain split primes anymore: one deals with more general sets S called r-covers. For ease of reference, we summarize the various abelian Stark conjectures that have been stated in the following table.

Table 1. The abelian Stark conjecture in the number field case.

	Classical (the set S contains split primes)	**Extended** (the set S is an r-cover)
Rank one S-version	Stark, Conjecture 1 of [24] Tate, Conjecture 2.2, page 89 of [26]	Erickson-Stark Conjecture 4.1 of [12]
Any order of vanishing (S,T)-version	Rubin Conjecture B' of [23]	Emmons-Popescu Conjecture 3.8 of [11]

Any conjecture in the column having as title "Classical" will be referred to loosely as *the classical abelian Stark conjecture*, whereas any conjecture in the column having as title "Extended" will be referred to as *the extended abelian Stark conjecture*. In his original paper [24], Stark stated the classical abelian rank one Stark conjecture only for a split prime which is archimedean. Tate's formulation of [26] includes the case where the split prime is finite as well. Conjecture B' of [23] is usually known under the name *the Rubin-Stark conjecture*. The term *rank* r is just another way of saying *order of vanishing* r. The Emmons-Popescu conjecture encompasses all the other conjectures contained in the table.

The author investigated numerically and theoretically the extended abelian *rank one* Stark conjecture in [28], [29], and [27]. As for the classical abelian Stark conjecture the rank one situation is special, and we think, should be studied separately. The algebra is less technically involved in this case, and everything is more transparent. One has both a S and (S,T)-version of the conjecture, where T is another finite set of primes disjoint from S and satisfying some properties recalled below. Among other things, the introduction of the auxiliary set of primes T and the associated modified (S,T)-imprimitive L-functions give a convenient way of dealing with the abelian condition of the S-version. Both these versions turn out to be equivalent. For the classical abelian rank one Stark conjecture, this is explained in Rubin's original paper [23]. For the extended abelian rank one Stark conjecture this equivalence is not in print anywhere, as far as we know, but can be proven in exactly the same way as in the classical case. In the higher order of vanishing case, the situation is not as satisfactory. There is a (S,T)-version, but no S-version is known to be equivalent to the (S,T)-version. In the classical case where S contains enough split primes, there is a conjecture of Popescu (Conjecture $C(K/k,S,r)$ in [19]) which gives such a S-version, but it is equivalent to the (S,T)-version only under certain hypotheses: see Theorem 5.5.1 of [19]. Moreover, from a computational point of view the rank one situation is more tractable than the higher order of vanishing situation. In fact, in the case where the split primes are infinite in the statement of the Rubin-Stark conjecture and the order of vanishing is at least two, it is not even known how to test the conjecture numerically.

In this paper, we shall work almost exclusively with the (S,T)-version of the extended abelian Stark conjecture, that is the Emmons-Popescu conjecture, even in the rank one case. As mentioned earlier, it includes all the other conjectures in the table as particular cases.

The goal of the following work is to show that the equivariant Tamagawa number conjecture implies the extended abelian Stark conjecture. In fact, we show that a stronger statement holds true. In [28] and [27], a Question was studied in the rank one case and here, we state a new conjecture, namely Conjecture 4.7, which lies, in some sense, between this Question and the rank one Emmons-Popescu conjecture. It is straightforward to extend this new conjecture to the higher order of vanishing situation which we do in Conjecture 4.16 and it is also clear that it implies the Emmons-Popescu conjecture. We then show that Conjecture 4.16 is implied by the equivariant Tamagawa number conjecture.

In particular, this gives the first proof of the Emmons-Popescu conjecture when the base field is $\mathbb{Q}$, since the equivariant Tamagawa number conjecture is known in this case due to the efforts of Burns and Greither in [8] and also of Flach in [13].

The paper is subdivided as follows. In §2, we review Tate sequences and in §3 we state the equivariant Tamagawa number conjecture as formulated by Burns and Flach in [7]. Particularly important for us is the reformulation of the equivariant Tamagawa number conjecture contained in §3.4 which is due to Burns. In §4, we recall the statement of the Emmons-Popescu conjecture and we state a stronger conjecture (Conjecture 4.16) which will be the main object of study of this paper. We also study the connection between this stronger conjecture and a Question that was studied in [27] and [28]. In §5, we present a useful reduction. We point out here that even if one is only interested in the rank one situation, this reduction forces the consideration of the higher order of vanishing case as well. We go on in §6 with showing that the equivariant Tamagawa number conjecture implies Conjecture 4.16 following the strategy used by Burns in [6] in the case of the Rubin-Stark conjecture. This is our main theorem which we state in §6.3. We end this paper with §7 where we state some explicit consequences of our work when the base field is $\mathbb{Q}$.

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1.1. **Basic notation.** If K/k is a finite abelian extension of number fields with Galois group G, then the symbol $S(K/k)$ denotes the finite set of places of k consisting of the archimedean places and the ones which ramify in K/k. The symbol S will typically stand for a finite set of primes of k containing $S(K/k)$. The set of places of K lying above places in S is denoted by S_K. We let $E_{K,S}$ be the group of S-units of the top field. Following [26], we let $Y_{K,S}$ denote the free abelian group on the places in S_K. It is acted upon by G in the usual way; hence, it is a $\mathbb{Z}[G]$-module. As usual, when we view $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module, the action of G on $\mathbb{Z}$ is always the trivial one. We have a surjective $\mathbb{Z}[G]$-module morphism $s : Y_{K,S} \longrightarrow \mathbb{Z}$ defined by $s(w) = 1$ for all $w \in S_K$. The kernel of this map is denoted by $X_{K,S}$. This leads to the following important short exact sequence of $\mathbb{Z}[G]$-modules

$$\xi_1 : 0 \longrightarrow X_{K,S} \longrightarrow Y_{K,S} \stackrel{s}{\longrightarrow} \mathbb{Z} \longrightarrow 0. \tag{1}$$

Throughout this paper, we will omit the tensor product symbol when we extend scalars. For instance, if M is a $\mathbb{Z}[G]$-module, then we simply write $\mathbb{C}M$ for the $\mathbb{C}[G]$-module $\mathbb{C} \otimes_{\mathbb{Z}} M$.

The Dirichlet logarithm $\lambda = \lambda_{K/k,S} : E_{K,S} \longrightarrow \mathbb{C}Y_{K,S}$ is defined by the formula

$$\lambda(u) = -\sum_{w \in S_K} \log |u|_w \cdot w,$$

where we use the usual normalization for the absolute values $|\cdot|_w$ so that the product formula holds. We would like to bring the negative sign to the attention of the reader. It is simple to check that λ is a $\mathbb{Z}[G]$-module morphism and that it takes values in $\mathbb{C}X_{K,S}$, precisely because of the product formula. The Dirichlet S-unit theorem implies that after extending this map to $\mathbb{C}E_{K,S}$ by $\mathbb{C}$-linearity, one gets an isomorphism of $\mathbb{C}[G]$-modules

$$\lambda : \mathbb{C}E_{K,S} \xrightarrow{\simeq} \mathbb{C}X_{K,S}.$$

For ease of writing, we will typically write

$$S = \{v_0, v_1, \ldots, v_n\},$$

so that $|S| = n+1$, and for each $i = 0, \ldots, n$, we fix a place w_i of K lying above v_i. If v is a place of k, we shall denote its decomposition group by G_v and we will abbreviate $G_i := G_{v_i}$. For $u \in E_{K,S}$ one has

$$\lambda(u) = -\sum_{w \in S_K} \log |u|_w \cdot w = \sum_{i=0}^{n} \ell_i(u) \cdot w_i,$$

where

$$\ell_i(u) = -\frac{1}{|G_i|} \sum_{\sigma \in G} \log |u^{\sigma}|_{w_i} \cdot \sigma^{-1} \in \mathbb{C}[G].$$

Each of these maps $\ell_i : E_{K,S} \longrightarrow \mathbb{C}[G]$ is a $\mathbb{Z}[G]$-module morphism and can also be extended by $\mathbb{C}$-linearity to a $\mathbb{C}[G]$-module morphism $\ell_i : \mathbb{C}E_{K,S} \longrightarrow \mathbb{C}[G]$.

Given a character $\chi \in \widehat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^{\times})$, we denote its associated S-imprimitive L-function by $L_{K/k,S}(s,\chi)$. For each character $\chi \in \widehat{G}$, we have the usual idempotent

$$e_{\chi} = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathbb{C}[G],$$

and one gathers the S-imprimitive L-functions into one equivariant S-imprimitive L-function in the usual way:

$$\theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S}(s,\chi) \cdot e_{\overline{\chi}}.$$

If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a meromorphic function, its first non-vanishing Taylor coefficient at $s = 0$ will be denoted by $f^*(0)$. We also set

$$\theta^*_{K/k,S}(0) = \sum_{\chi \in \widehat{G}} L^*_{K/k,S}(0,\chi) \cdot e_{\overline{\chi}}.$$

Roughly speaking, both the (extended) abelian Stark conjecture and the equivariant Tamagawa number conjecture predict a link between $E_{K,S}$ and $\theta^*_{K/k,S}(0)$ via the Dirichlet logarithm λ (or at least $e \cdot \theta^*_{K/k,S}(0)$ for some idempotent e to be made precise later).

In this paper, we shall work almost exclusively with the (S,T)-version of the extended abelian Stark conjecture even if the order of vanishing is one. Hence, we now introduce an auxiliary finite set of finite primes T satisfying $S \cap T = \varnothing$. We let $E_{K,S,T}$ be the group of (S,T)-units of K. The S_K-class group of K is denoted by $Cl_{K,S}$ and the (S_K, T_K)-class group by $Cl_{K,S,T}$. If the readers need to review these notions, they can consult [20]. In order to link these S and (S,T)-objects together via an exact sequence, one introduces the following $\mathbb{Z}[G]$-module

$$\mathbb{F}_T^{\times} = \bigoplus_{\mathfrak{P} \in T_K} \mathbb{F}_{\mathfrak{P}}^{\times},$$

where $\mathbb{F}_{\mathfrak{P}}$ denotes the residue field of K at $\mathfrak{P}$. We then have an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow E_{K,S,T} \longrightarrow E_{K,S} \longrightarrow \mathbb{F}_T^{\times} \longrightarrow Cl_{K,S,T} \longrightarrow Cl_{K,S} \longrightarrow 0, \tag{2}$$

from which, one deduces that $Cl_{K,S,T}$ is finite and $E_{K,S,T}$ is a subgroup of $E_{K,S}$ of finite index. Therefore, the Dirichlet logarithm still induces an isomorphism of $\mathbb{C}[G]$-modules

$$\lambda : \mathbb{C}E_{K,S,T} \xrightarrow{\simeq} \mathbb{C}X_{K,S}.$$

Following the general yoga of the (S,T)-version of Stark's conjecture, we now modify the equivariant S-imprimitive L-function as follows. We let

$$\delta_T(s) = \prod_{\mathfrak{p}\in T}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})^{1-s}),$$

where $\sigma_{\mathfrak{p}}$ denotes the Frobenius automorphism of $\mathfrak{p}$ in K/k. The equivariant (S,T)-imprimitive L-function is defined to be

$$\theta_{K/k,S,T}(s) := \delta_T(s)\cdot\theta_{K/k,S}(s).$$

Since we will be strictly concerned with the special value $s=0$, we shall write simply δ_T rather than $\delta_T(0)$. Remark that the order of vanishing at $s=0$ of $L_{K/k,S}(s,\chi)$ and of $\overline{\chi}(\delta_T(s))\cdot L_{K/k,S}(s,\chi)$ are the same for all $\chi\in\widehat{G}$, since $\overline{\chi}(\delta_T)\neq 0$.

For future use, we record here the following well-known proposition. If R is a commutative ring with 1 and M a finitely generated R-module, we denote its first Fitting ideal by $\mathrm{Fit}_R(M)$.

Proposition 1.1. *The $\mathbb{Z}[G]$-module $\mathbb{F}_T^\times$ is cohomologically trivial. Furthermore,*

$$\mathrm{Fit}_{\mathbb{Z}[G]}(\mathbb{F}_T^\times) = \mathbb{Z}[G]\cdot\delta_T.$$

Proof. For the purpose of this proof, given a prime $\mathfrak{p}\in T$ let

$$M_{\mathfrak{p}} := \bigoplus_{\mathfrak{P}|\mathfrak{p}}\mathbb{F}_{\mathfrak{P}}^\times.$$

One has a short exact sequence of $\mathbb{Z}[G]$-modules

$$0\longrightarrow\mathbb{Z}[G]\longrightarrow\mathbb{Z}[G]\longrightarrow M_{\mathfrak{p}}\longrightarrow 0,$$

where the first arrow is multiplication by $1-\sigma_{\mathfrak{p}}^{-1}\cdot\mathbb{N}(\mathfrak{p})$, and the second one is obtained as follows: for $\mathfrak{P}|\mathfrak{p}$, choose a primitive root $\zeta_{\mathfrak{P}}+\mathfrak{P}\in\mathbb{F}_{\mathfrak{P}}^\times$. The second map is then given by

$$\sigma\mapsto(\zeta_{\mathfrak{P}}^{\sigma}+\mathfrak{P}^{\sigma})_{\mathfrak{P}|\mathfrak{p}},$$

for $\sigma\in G$. We deduce that $M_{\mathfrak{p}}$ is cohomologically trivial and

$$\mathrm{Fit}_{\mathbb{Z}[G]}\left(M_{\mathfrak{p}}\right) = (1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p}))\cdot\mathbb{Z}[G].$$

The corresponding properties for $\mathbb{F}_T^\times$ follow immediately, since

$$\mathbb{F}_T^\times = \bigoplus_{\mathfrak{p}\in T}M_{\mathfrak{p}}.$$

□

2. Review of Tate sequences

In this section, we review Tate sequences, since they are essential for the statement of the equivariant Tamagawa number conjecture. Before explaining Tate sequences, we start in §2.1 with a brief overview of the interpretation of $\mathrm{Ext}_R^2(M,N)$ in terms of equivalence classes of 2-extensions of M by N which is due to Yoneda.

Throughout this section if M is a $\mathbb{Z}[G]$-module, then $H^m(G,M)$ denotes the usual cohomology groups for $m\geq 0$. The modified cohomology groups (or the Tate cohomology groups) will be denoted by $\widehat{H}^m(G,M)$ for $m\in\mathbb{Z}$. Let us recall that we have $H^m(G,M)=\widehat{H}^m(G,M)$ for $m\geq 1$.

2.1. Ext^2 **and Yoneda 2-extensions.** Let R be any commutative ring with 1. Let M and N be R-modules and let us fix a R-projective resolution of M:

$$P^\bullet : \ldots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \longrightarrow 0.$$

By definition, the groups $\text{Ext}^n_R(M,N)$ are obtained by applying the functor $\text{Hom}_R(\_\,,N)$ to the projective resolution $P^\bullet$ and then by taking cohomology. More precisely, after applying $\text{Hom}_R(\_\,,N)$ to $P^\bullet$, we obtain the following cochain complex

$$\text{Hom}_R(P_0,N) \xrightarrow{\partial_1^*} \text{Hom}_R(P_1,N) \xrightarrow{\partial_2^*} \text{Hom}_R(P_2,N) \xrightarrow{\partial_3^*} \ldots$$

One defines $\text{Ext}^0_R(M,N) := \text{Ker}(\partial_1^*)$ and for $n \geq 1$

$$\text{Ext}^n_R(M,N) := \text{Ker}(\partial_{n+1}^*)/\text{Im}(\partial_n^*).$$

In [30], Yoneda gave another interpretation of the groups $\text{Ext}^n_R(M,N)$ in terms of equivalence classes of n-extensions of M by N. In this section, we only recall what we strictly need later and in order to simplify notations, we work only with $n = 2$. One can have a look at [17] for more details. A 2-extension of M by N is by definition an exact sequence

$$\xi : 0 \longrightarrow N \longrightarrow E_0 \longrightarrow E_1 \longrightarrow M \longrightarrow 0.$$

Given two 2-extensions ξ and ξ' of M by N, one would like to say that they are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccccc}
\xi : 0 & \longrightarrow & N & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
\xi' : 0 & \longrightarrow & N & \longrightarrow & E_0' & \longrightarrow & E_1' & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

but this relation is not symmetric (in contrast with the case $n = 1$). The equivalence relation it generates will be denoted by $\sim$ and the set of equivalence classes of 2-extensions by $\text{YExt}^2_R(M,N)$. One can show that $\text{YExt}^2_R(M,N)$ has the structure of an abelian group. Moreover, $\text{YExt}^2_R(\_\,,\_)$ is a bifunctor and there is a natural equivalence

$$\text{Ext}^2_R(\_\,,\_) \xrightarrow{\simeq} \text{YExt}^2_R(\_\,,\_).$$

(In fact, an analogous result holds for all values of n, not only $n = 2$.)

We recall only two things about this natural equivalence when $n = 2$. First, we recall how the map

$$\theta : \text{Ext}^2_R(M,N) \xrightarrow{\simeq} \text{YExt}^2_R(M,N) \tag{3}$$

is defined. Starting with $c = f + \text{Im}(\partial_2^*) \in \text{Ext}^2_R(M,N)$, where $f \in \text{Ker}(\partial_3^*)$, let us consider the exact sequence

$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

coming from the projective resolution $P^\bullet$, where $K_2 = \text{Ker}(\partial_1) \simeq \text{Coker}(\partial_3)$. Since $f \in \text{Ker}(\partial_3^*)$, it induces a map $f : K_2 \longrightarrow N$. Let Ω be the push-out of the diagram

$$\begin{array}{ccc}
K_2 & \longrightarrow & P_1 \\
{\scriptstyle f}\downarrow & & \\
N & &
\end{array}$$

Because of standard properties of the push-out, we obtain the following commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & \Omega & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where the bottom row is exact; hence, it is a 2-extension of M by N. Let us denote it by ξ and its class in $\mathrm{YExt}^2_R(M,N)$ by $[\xi]$. The map θ of (3) above is given by

$$\theta(c) = [\xi].$$

The second thing we want to recall is the description of the map

$$\iota_* : \mathrm{YExt}^2_R(M,N) \longrightarrow \mathrm{YExt}^2_R(M,N') \tag{4}$$

given a morphism $\iota : N \longrightarrow N'$. Starting with a 2-extension

$$0 \longrightarrow N \longrightarrow E_0 \longrightarrow E_1 \longrightarrow M \longrightarrow 0,$$

let Ω be the push-out of the diagram

$$\begin{array}{ccc} N & \longrightarrow & E_0 \\ {\scriptstyle\iota}\big\downarrow & & \\ N' & & \end{array}$$

Standard properties of the push-out give the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\iota}\big\downarrow & & \big\downarrow & & \big\| & & \big\| & & \\ 0 & \longrightarrow & N' & \longrightarrow & \Omega & \longrightarrow & E_1 & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where the second row is exact; hence, it is a 2-extension of M by N'. We denote it by ξ' and the 2-extension we started with by ξ. The map ι_* of (4) is given by

$$\iota_*([\xi]) = [\xi'].$$

2.2. **Tate sequences.** As before, K/k is a finite abelian extension of number fields with Galois group G and S a finite set of places of k containing $S(K/k)$. Given a place w of K, we denote the completion of K at w by K_w and the group of local units by U_w. We also let

$$\mathcal{J}_{K,S} = \prod_{w \in S_K} K_w^\times \times \prod_{w \notin S_K} U_w,$$

be the group of S-ideles of K. Moreover, we denote the idele class group by C_K (whereas the ideal class group is denoted by Cl_K). In this section, we make the following hypothesis:

Hypothesis 2.1. *The S-class group is trivial: $Cl_{K,S} = 1$.*

Under Hypothesis 2.1, we have the following short exact sequence of $\mathbb{Z}[G]$-modules

$$\xi_2 : 0 \longrightarrow E_{K,S} \longrightarrow \mathcal{J}_{K,S} \xrightarrow{t} C_K \longrightarrow 0,$$

which one can compare with the short exact sequence (1). Global class field theory relates the cohomology of $\mathbb{Z}$ to the cohomology of the idele class group C_K. Local class field theory relates the cohomology of $\mathbb{Z}$ to the cohomology of $K_w^\times$ which in turn can be used to relate the cohomology of $Y_{K,S}$ to the cohomology of $\mathcal{J}_{K,S}$. Assuming Hypothesis 2.1, Tate showed in [25] how one can relate the cohomology of $E_{K,S}$ to the cohomology of $X_{K,S}$ using the usual compatibility relation between local and global class field theory (see also [26], chapter II, §5). In the process, he exhibited a canonical $\alpha_3 \in \widehat{H}^2(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, E_{K,S}))$. Since $X_{K,S}$ is $\mathbb{Z}$-free (see Lemma 2.3 below), one has an isomorphism

$$\widehat{H}^2(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, E_{K,S})) \simeq \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S}).$$

The image of α_3 under this isomorphism will be denoted by $c_{K/k,S}$: it is called the fundamental class (or the Tate class). Using the interpretation of the abelian group $\mathrm{Ext}^2_{\mathbb{Z}[G]}(M,N)$ as the set of equivalence classes of 2-extensions of M by N recalled in §2.1, the $c_{K/k,S}$ leads to a 2-extension of $X_{K,S}$ by $E_{K,S}$:

$$0 \longrightarrow E_{K,S} \longrightarrow \Delta_0 \longrightarrow \Delta_1 \longrightarrow X_{K,S} \longrightarrow 0.$$

One can also assume that both Δ_0 and Δ_1 are finitely generated over $\mathbb{Z}[G]$, Δ_0 is cohomologically trivial, and Δ_1 is $\mathbb{Z}[G]$-projective (even $\mathbb{Z}[G]$-free, but this does not play a role in the sequel). This leads to the following definition:

Definition 2.2. Let K/k be a finite abelian extension and let S be a finite set of primes of k satisfying $S \supseteq S(K/k)$. Suppose also that $Cl_{K,S} = 1$. A Tate sequence is a 2-extension of $X_{K,S}$ by $E_{K,S}$

$$0 \longrightarrow E_{K,S} \longrightarrow \Delta_0 \longrightarrow \Delta_1 \longrightarrow X_{K,S} \longrightarrow 0,$$

whose corresponding class in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S})$ is $c_{K/k,S}$ and such that Δ_0 is cohomologically trivial, Δ_1 is $\mathbb{Z}[G]$-projective and both Δ_0 and Δ_1 are finitely generated $\mathbb{Z}[G]$-modules.

We end this section with two lemmas which will be used repeatedly in this paper.

Lemma 2.3. *Let M and N be $\mathbb{Z}[G]$-modules. If M is $\mathbb{Z}$-free, then we have an isomorphism*

$$H^m(G, \mathrm{Hom}_{\mathbb{Z}}(M, N)) \xrightarrow{\simeq} \mathrm{Ext}^m_{\mathbb{Z}[G]}(M, N),$$

for all integers $m \geq 0$.

Proof. Let

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{5}$$

be a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$. Remark that for $m \geq 1$, one has

$$H^m(G, \mathrm{Hom}_{\mathbb{Z}}(M, N)) = \mathrm{Ext}^m_{\mathbb{Z}[G]}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(M, N)).$$

If P is a projective $\mathbb{Z}[G]$-module, then $P \otimes_{\mathbb{Z}} M$ is also a projective $\mathbb{Z}[G]$-module, since it is both cohomologically trivial and $\mathbb{Z}$-free. Since M is $\mathbb{Z}$-free, it is also $\mathbb{Z}$-flat; thus, the functor $\_ \otimes_{\mathbb{Z}} M$ is exact. Therefore, after applying this last functor to the projective resolution (5), we obtain a $\mathbb{Z}[G]$-projective resolution of M:

$$\ldots \longrightarrow P_2 \otimes_{\mathbb{Z}} M \longrightarrow P_1 \otimes_{\mathbb{Z}} M \longrightarrow P_0 \otimes_{\mathbb{Z}} M \longrightarrow M \longrightarrow 0. \tag{6}$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\_, \mathrm{Hom}_{\mathbb{Z}}(M, N))$ to (5), we obtain the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_0, \mathrm{Hom}_{\mathbb{Z}}(M, N)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, \mathrm{Hom}_{\mathbb{Z}}(M, N)) \longrightarrow \ldots$$

from which the groups $H^m(G, \mathrm{Hom}_{\mathbb{Z}}(M, N))$ are obtained by taking its cohomology. On the other hand, applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\_, N)$ to (6), we obtain the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_0 \otimes_{\mathbb{Z}} M, N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_1 \otimes_{\mathbb{Z}} M, N) \longrightarrow \ldots$$

from which the groups $\mathrm{Ext}^m_{\mathbb{Z}[G]}(M, N)$ are obtained by taking its cohomology. We can conclude the desired result using the isomorphism of $\mathbb{Z}[G]$-modules

$$\mathrm{Hom}_{\mathbb{Z}[G]}(A, \mathrm{Hom}_{\mathbb{Z}}(B, C)) \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(A \otimes_{\mathbb{Z}} B, C),$$

valid for any $\mathbb{Z}[G]$-modules A, B, and C.

□

Lemma 2.4. *Let*

$$0 \longrightarrow E_{K,S} \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow X_{K,S} \longrightarrow 0, \tag{7}$$

be a 2-extension of $X_{K,S}$ by $E_{K,S}$ whose class corresponds to $c_{K/k,S} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S})$. If Q_1 is $\mathbb{Z}[G]$-projective, then Q_0 is cohomologically trivial.

Proof. We just sketch one possible proof. One can break the short exact sequence (7) into two short exact sequences

$$0 \longrightarrow E_{K,S} \longrightarrow Q_0 \longrightarrow M \longrightarrow 0, \tag{8}$$

and

$$0 \longrightarrow M \longrightarrow Q_1 \longrightarrow X_{K,S} \longrightarrow 0. \tag{9}$$

Since Q_1 is $\mathbb{Z}[G]$-projective, the $\mathbb{Z}[G]$-module M is $\mathbb{Z}$-free. It follows from Lemma 2.3 that we obtain $\alpha \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, E_{K,S}))$ corresponding to the class of (8) in $\mathrm{Ext}^1_{\mathbb{Z}[G]}(M, E_{K,S})$. Similarly, we obtain a $\beta \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, M))$ from (9). The cup product induces maps

$$\widehat{H}^m(G, X_{K,S}) \xrightarrow{\smile\beta} \widehat{H}^{m+1}(G, M) \xrightarrow{\smile\alpha} \widehat{H}^{m+2}(G, E_{K,S}),$$

whose composition is cup product with α_3. Since Q_1 is $\mathbb{Z}[G]$-projective, cup product with α is an isomorphism. Cup product with α_3 being an isomorphism, it follows that cup product with β is also an isomorphism from which we deduce that $\widehat{H}^m(G, Q_0) = 0$ for all $m \in \mathbb{Z}$. The same argument would work if one replaces G by a subgroup G'; hence, Q_0 is cohomologically trivial as we wanted to show. □

3. The equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(K)), \mathbb{Z}[G])$

The goal of this section is to explain a reformulation of the equivariant Tamagawa number conjecture due to Burns which will be particularly useful for us (Proposition 3.10). We start in §3.1 by recalling two different ways of viewing fractional ideals. In §3.2 we go on with some reminders on the Det functor of Grothendieck-Knudsen-Mumford. In §3.3, we recall the statement of the equivariant Tamagawa number conjecture as formulated in the abelian setting by Burns and Flach in [7] and at last, in §3.4, we explain a more concrete reformulation due to Burns of the equivariant Tamagawa number conjecture which takes into account the auxiliary set of finite primes T showing up in the statement of the (S, T)-version of the abelian Stark conjecture.

Throughout this section, we let R be a commutative ring with 1. In fact, for the applications we have in mind in this paper, R is either $\mathbb{C}$, $\mathbb{Z}[G]$ or $\mathbb{C}[G]$.

3.1. **Preliminaries on fractional ideals.** Recall that a fractional ideal of R is a R-module $\mathfrak{a}$ contained in the total ring of fractions $Q(R)$ such that there exists a non-zero divisor $x \in R$ satisfying $x\mathfrak{a} \subseteq R$. The fractional ideal $\mathfrak{a}$ is said to be invertible if there exists another fractional ideal $\mathfrak{b}$ such that

$$\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a} = R.$$

The set of invertible fractional ideals I_R form a group under multiplication of fractional ideals. One defines the subgroup P_R of I_R consisting of principal ideals. These are of the form Rx for some $x \in Q(R)^\times$. The class group of R is by definition the quotient

$$Cl_R := I_R/P_R.$$

When K is a number field and $R = O_K$, we get back the usual class group Cl_K.

There is another way of seeing the group of invertible fractional ideals of R as follows. Recall that if P is a finitely generated projective R-module then $P_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of finite rank for all prime ideals $\mathfrak{p}$ of R. This leads to the rank function

$$rk_P : \mathrm{Spec}(R) \longrightarrow \mathbb{Z},$$

defined as $\mathfrak{p} \mapsto rk_P(\mathfrak{p}) = \mathrm{rank}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$. It is known that the rank function is continuous or equivalently locally constant.

Let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules of constant rank one whose morphisms are morphisms of R-modules. It turns out that the isomorphism classes of objects in $\mathcal{P}(R)$ form an abelian group denoted by Pic_R, the operation being induced by the tensor product. The neutral element is given by the isomorphism class of R. If $[P]$ denotes the isomorphism class of a finitely generated projective R-module P of constant rank one, then its inverse is given by $[\mathrm{Hom}_R(P, R)]$. Indeed, for every such P, one has an isomorphism given by the evaluation map

$$ev_P : P \otimes_R \mathrm{Hom}_R(P, R) \xrightarrow{\simeq} R, \tag{10}$$

defined as $m \otimes f \mapsto f(m)$.

If $\mathfrak{a}$ is an invertible fractional ideal, then one can show that $\mathfrak{a}$ is a finitely generated projective R-module of constant rank one. Therefore, one has a map $I_R \longrightarrow \mathrm{Pic}_R$ and in fact, this map induces an isomorphism of abelian groups

$$Cl_R \xrightarrow{\simeq} \mathrm{Pic}_R.$$

3.2. **The** Det **functor.** If P is a finitely generated projective R-module, then we will denote its rank function more briefly by r. If the rank function is constant, then it is clear what we mean by $\bigwedge_R^r P$. Otherwise, the R-module $\bigwedge_R^r P$ is the R-module such that locally one has

$$\left(\bigwedge_R^r P\right)_{\mathfrak{p}} = \bigwedge_{R_{\mathfrak{p}}}^{r(\mathfrak{p})} P_{\mathfrak{p}}.$$

One can associate to P a R-module $\det_R(P) \in \mathrm{Ob}(\mathcal{P}(R))$ as follows:

$$\det_R(P) := \bigwedge_R^r P.$$

Remark. If $R = \mathbb{Z}[G]$, where G is a finite abelian group, then it is known that there are only two idempotents in $\mathbb{Z}[G]$, namely 1 and 0; therefore, $\mathrm{Spec}(\mathbb{Z}[G])$ is connected and the rank function is constant for any finitely generated projective $\mathbb{Z}[G]$-module.

This defines a functor

$$\det_R : Proj(R) \longrightarrow \mathcal{P}(R),$$

where $Proj(R)$ is the category whose objects are finitely generated projective R-modules and the morphisms are the morphisms of R-modules. This functor satisfies certain properties and we now recall some of them.

(1) Given a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow 0,$$

where $P_i \in \mathrm{Ob}(Proj(R))$ for $i = 1, 2, 3$, one has

$$\det_R(P_2) \simeq \det_R(P_1) \otimes_R \det(P_3).$$

Indeed, since P_3 is projective, the short exact sequence splits; therefore $P_2 \simeq P_1 \oplus P_3$. Let us denote the rank functions by r_i for $i = 1, 2, 3$. A usual property of the wedge product implies

$$\bigwedge_R^{r_2} P_2 \simeq \bigoplus_{i=0}^{r_2} \left(\left(\bigwedge_R^{i} P_1 \right) \otimes_R \left(\bigwedge_R^{r_2 - i} P_3 \right) \right).$$

Noting that $r_2 = r_1 + r_3$, we obtain the desired identity

$$\bigwedge_R^{r_2} P_2 \simeq \bigwedge_R^{r_1} P_1 \otimes_R \bigwedge_R^{r_3} P_3$$

(2) This functor commutes with base change, so if we have a ring morphism $R \longrightarrow S$, then

$$S \otimes_R \det_R(P) \simeq \det_S(S \otimes_R M).$$

Indeed, this is clear from the usual property of the wedge product

$$S \otimes_R \bigwedge_R^r P \simeq \bigwedge_S^r S \otimes_R P,$$

combined with the fact that the rank function of $S \otimes_R P$ is given by the composition

$$\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R) \xrightarrow{r} \mathbb{Z},$$

where r is the rank function of P as a R-module.

For our purposes, $\mathcal{P}(R)$ will not be enough. In order to resolve a sign ambiguity, it was realized that the category of graded line bundles $\mathcal{P}gr(R)$ should be used instead of $\mathcal{P}(R)$. For this matter, see §2.5 of [7].

The objects of the category $\mathcal{P}gr(R)$ are pairs (P,α), where P is a finitely generated projective R-module of constant rank one and $\alpha : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}$ is a continuous function. Morphisms $f : (P,\alpha) \longrightarrow (Q,\beta)$ consist of a morphism of R-modules $f : P \longrightarrow Q$ such that whenever $\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p})$, then $f_{\mathfrak{p}} : P_{\mathfrak{p}} \longrightarrow Q_{\mathfrak{p}}$ is the zero function. One defines a tensor product in this category as follows:

$$(P,\alpha) \otimes (Q,\beta) := (P \otimes_R Q, \alpha+\beta).$$

The commutativity constraint is given by the isomorphism

$$(P,\alpha) \otimes (Q,\beta) \longrightarrow (Q,\beta) \otimes (P,\alpha),$$

defined locally for $m \in P_{\mathfrak{p}}$ and $n \in Q_{\mathfrak{p}}$ as

$$m \otimes n \mapsto (-1)^{\alpha(\mathfrak{p})\beta(\mathfrak{p})} n \otimes m.$$

One sets

$$(P,\alpha)^{-1} = (\mathrm{Hom}_R(P,R), -\alpha),$$

and $(P,\alpha)^{-1}$ is a right inverse via the isomorphism

$$(P,\alpha) \otimes (P,\alpha)^{-1} \xrightarrow{\simeq} (R,0),$$

induced by the evaluation map (10). A right inverse is considered as a left inverse via the commutativity constraint. We let $\mathcal{P}gr_{is}(R)$ be the category obtained from $\mathcal{P}gr(R)$ by considering isomorphisms only.

Let $\mathcal{E}$ be an exact category. (An example of an exact category is provided by $Proj(R)$.) If Ω is a set of morphisms of $\mathcal{E}$ containing all isomorphisms and closed under composition of morphisms, then we let $\mathcal{E}_\Omega$ be the subcategory of $\mathcal{E}$ where morphisms are restricted to the ones in Ω.

A determinant functor f on $(\mathcal{E},\Omega)$ with values in $\mathcal{P}gr_{is}(R)$ is a pair $f = (f_1, f_2)$ consisting of a functor

$$f_1 : \mathcal{E}_\Omega \longrightarrow \mathcal{P}gr_{is}(R)$$

and for every short exact sequence in $\mathcal{E}$

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

an isomorphism

$$f_2(\alpha,\beta) : f_1(B) \xrightarrow{\simeq} f_1(A) \otimes f_1(C) \tag{11}$$

satisfying a certain set of axioms listed in [18] (see also [3]).

Among the first determinant functors to be studied was the one of [18] which we now briefly recall. Consider the exact category $\mathcal{E} = Proj(R)$ consisting of finitely generated projective R-modules and where Ω consists of the isomorphisms. One defines a determinant functor $\mathrm{Det}_R = (\det_R, i) : (Proj(R),\Omega) \longrightarrow \mathcal{P}gr_{is}(R)$, by setting

$$\mathrm{Det}_R(P) = (\det_R(P), r)\,,$$

where r is the rank function of the R-module P and $i(\alpha,\beta)$ is the usual isomorphism.

It is simple to extend the Det functor to the exact category $\mathcal{E} = \mathcal{C}^b(Proj(R))$ consisting of bounded complexes of finitely generated projective R-modules and $\Omega = qis$ consists of quasi-isomorphisms. We remind the reader that a quasi-isomorphism between two cochain complexes is a morphism of cochain complexes which induces isomorphisms at the level of cohomology. If $B^\bullet$ is a bounded cochain complex of finitely generated projective R-modules, then one extends the Det_R functor by setting

$$\mathrm{Det}_R(B^\bullet) := \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_R(B_i)^{(-1)^i}.$$

Remark. Burns does not use the same normalization in [4] as in [6]. We are using here the one of [6] which is the same as the one of the original paper [18].

Recall that a cochain complex $C^\bullet$ is called perfect if there exists a bounded complex $B^\bullet$ of finitely generated projective R-modules and a quasi-isomorphism $B^\bullet \longrightarrow C^\bullet$. We let $\mathcal{C}^{perf}(R)$ denote the category whose objects are perfect complexes and morphisms are the usual morphisms of cochain complexes; it is an exact category. Let $\Omega = qis$ be the set of quasi-isomorphisms. Knudsen and Mumford showed in [18] that the Det functor can be extended to $(\mathcal{C}^{perf}(R), qis)$.

Remark. In this paper, we do not use the language of derived categories. Sometimes, it simplifies certain arguments, but we choose to adopt the pedestrian approach here. A derived category is not an abelian category anymore; hence the notion of short exact sequence does not make sense. This notion is replaced by distinguished triangles. The determinant functor $(\mathcal{C}^{perf}(R), qis) \longrightarrow \mathcal{P}gr_{is}(R)$ factors through the derived category $\mathcal{D}^{perf}(R)$ and then we have to face the question of how this determinant functor behaves with respect to distinguished triangles. We will not need to deal with these issues in this paper, but the interested reader could have a look at [3].

3.3. **The equivariant Tamagawa number conjecture.** In this section, we have to start with a Tate sequence; hence, we shall make Hypothesis 2.1 throughout. Let

$$0 \longrightarrow E_{K,S} \longrightarrow \Delta_0 \xrightarrow{\delta} \Delta_1 \longrightarrow X_{K,S} \longrightarrow 0,$$

be such a Tate sequence. After tensoring with $\mathbb{C}$, we obtain an exact sequence of $\mathbb{C}[G]$-modules

$$0 \longrightarrow \mathbb{C}E_{K,S} \longrightarrow \mathbb{C}\Delta_0 \xrightarrow{\delta} \mathbb{C}\Delta_1 \longrightarrow \mathbb{C}X_{K,S} \longrightarrow 0.$$

This last short exact sequence breaks up into the two following ones

$$0 \longrightarrow \mathbb{C}E_{K,S} \longrightarrow \mathbb{C}\Delta_0 \longrightarrow \mathbb{C}\mathrm{Im}(\delta) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbb{C}\mathrm{Im}(\delta) \longrightarrow \mathbb{C}\Delta_1 \longrightarrow \mathbb{C}X_{K,S} \longrightarrow 0.$$

These two short exact sequences of $\mathbb{C}[G]$-modules combined with (11) imply that

$$\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta_0) \simeq \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}E_{K,S}) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta)), \tag{12}$$

and

$$\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta_1) \simeq \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta)) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}X_{K,S}). \tag{13}$$

Let $\Delta^\bullet$ be the cochain complex

$$\Delta^\bullet : \ldots \longrightarrow 0 \longrightarrow \Delta_0 \xrightarrow{\delta} \Delta_1 \longrightarrow 0 \longrightarrow \ldots$$

where Δ_0 is placed in degree 0. Remark that $H^0(\Delta^\bullet) = E_{K,S}$ and $H^1(\Delta^\bullet) = X_{K,S}$.

One lets

$$\vartheta_{\Delta^\bullet,\lambda} : \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta^\bullet) \xrightarrow{\simeq} (\mathbb{C}[G], 0)$$

be the isomorphism obtained via the composition of the following isomorphisms

$$\begin{aligned}
\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta^\bullet) &\xrightarrow{=} \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta_0) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\Delta_1)^{-1} \\
&\xrightarrow{\simeq} \big(\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}E_{K,S}) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta))\big) \otimes \big(\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta)) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}X_{K,S})\big)^{-1} \\
&\xrightarrow{\simeq} \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}E_{K,S}) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}X_{K,S})^{-1} \\
&\xrightarrow{\simeq} \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}X_{K,S}) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}X_{K,S})^{-1} \\
&\xrightarrow[\simeq]{ev} (\mathbb{C}[G], 0),
\end{aligned}$$

where the first arrow is by definition of the Det functor, the second arrow comes from (12) and (13), the third from the evaluation map

$$\mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta)) \otimes \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}\mathrm{Im}(\delta))^{-1} \xrightarrow[\simeq]{ev} (\mathbb{C}[G], 0),$$

and the fourth one is $\mathrm{Det}_{\mathbb{C}[G]}(\lambda) \otimes id$.

Lemma 3.1. *The complex $\Delta^\bullet$ is an object in $\mathcal{C}^{perf}(\mathbb{Z}[G])$.*

Proof. Since Δ_0 is cohomologically trivial, there exists a short exact sequence of the form

$$\ldots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} \Delta_0 \longrightarrow 0.$$

Then, one obtains a bounded cochain complex of finitely generated projective $\mathbb{Z}[G]$-modules

$$B^\bullet : \ldots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta\circ\varepsilon} \Delta_1 \longrightarrow 0 \longrightarrow \ldots$$

which fits into the following commutative diagram

$$\begin{array}{ccccccccccccc} B^\bullet : \ldots & \longrightarrow & 0 & \longrightarrow & P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta\circ\varepsilon} & \Delta_1 & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & \| & & & & \\ \Delta^\bullet : \ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Delta_0 & \xrightarrow{\delta} & \Delta_1 & \longrightarrow & 0 & \longrightarrow & \ldots \end{array}$$

It is simple to check that this morphism of cochain complexes $B^\bullet \longrightarrow \Delta^\bullet$ is in fact a quasi-isomorphism. □

Remark. Because of the last lemma, it makes sense to apply the $\mathrm{Det}_{\mathbb{Z}[G]}$ functor to the complex $\Delta^\bullet$ and evaluate $\vartheta_{\Delta^\bullet,\lambda}(\mathrm{Det}_{\mathbb{Z}[G]}(\Delta^\bullet))$.

Conjecture 3.2 (The equivariant Tamagawa number conjecture or the leading term conjecture)**.** *Always under Hypothesis 2.1, one has in $\mathcal{P}gr(\mathbb{C}[G])$ the following equality*

$$\vartheta_{\Delta^\bullet,\lambda}\left(\mathrm{Det}_{\mathbb{Z}[G]}(\Delta^\bullet)\right) = (\mathbb{Z}[G]\cdot\theta^*_{K/k,S}(0), 0).$$

Remark. It is known that this conjecture does not depend on the choice of the Tate sequence and hence on $\Delta^\bullet$. See remark 6.2 in [6].

3.4. **A reformulation of the equivariant Tamagawa number conjecture.** In this section, we explain a concrete reformulation of the equivariant Tamagawa number conjecture which also takes into account the auxiliary set of finite primes T which appears in the (S,T)-version of the abelian Stark conjecture. The results of this section are due to Burns (see Proposition 7.2 of [6]).

3.4.1. *Preliminaries.* If M is a $\mathbb{Z}[G]$-module, then we let

$$M^* = \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]),$$

that is, M^* is the dual of M in the category of $\mathbb{Z}[G]$-modules.

If $\varphi \in M^*$, then for any integer $r \geq 1$ it induces a $\mathbb{Z}[G]$-module morphism

$$\tilde{\varphi} : \bigwedge_{\mathbb{Z}[G]}^{r} M \longrightarrow \bigwedge_{\mathbb{Z}[G]}^{r-1} M,$$

defined by

$$m_1\wedge\ldots\wedge m_r \mapsto \sum_{i=1}^{r}(-1)^{i+1}\varphi(m_i)m_1\wedge\ldots\wedge m_{i-1}\wedge m_{i+1}\wedge\ldots\wedge m_r.$$

If $\varphi_1,\ldots,\varphi_k \in M^*$, then iterating this process gives a $\mathbb{Z}[G]$-module morphism

$$\bigwedge_{\mathbb{Z}[G]}^{k} M^* \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}\left(\bigwedge_{\mathbb{Z}[G]}^{r} M, \bigwedge_{\mathbb{Z}[G]}^{r-k} M\right),$$

defined by $\varphi_1\wedge\ldots\wedge\varphi_k \mapsto \tilde{\varphi}_k\circ\ldots\circ\tilde{\varphi}_1$. When $k=r$, we obtain a map

$$\bigwedge_{\mathbb{Z}[G]}^{r} M^* \longrightarrow \left(\bigwedge_{\mathbb{Z}[G]}^{r} M\right)^*$$

and a simple computation shows that

$$\varphi_1\wedge\ldots\wedge\varphi_r(m_1\wedge\ldots\wedge m_r) = \det(\varphi_i(m_j)).$$

In this paper, we will have to deal with different kinds of $\mathbb{Z}[G]$-modules. For example, by a lattice, we shall mean a finitely generated $\mathbb{Z}[G]$-module which is $\mathbb{Z}$-free. The $\mathbb{Z}[G]$-module $E_{K,S,T}$ is an example of a lattice.

Another kind of important $\mathbb{Z}[G]$-modules are the finitely generated projective $\mathbb{Z}[G]$-modules. If M is a finitety generated projective R-module, then $(M^*)^* \simeq M$.

If M and N are $\mathbb{C}[G]$-modules and $f \in \mathrm{Hom}_{\mathbb{C}[G]}(M,N)$, then given $\chi \in \widehat{G}$, we let f^χ be the corresponding morphism of $\mathbb{C}$-vector spaces

$$f^\chi : M \cdot e_\chi \longrightarrow N \cdot e_\chi.$$

If M and N are $\mathbb{Z}[G]$-modules and $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(M,N)$, then we let f^χ denote the morphism of $\mathbb{C}$-vector spaces

$$f^\chi : \mathbb{C}M \cdot e_\chi \longrightarrow \mathbb{C}N \cdot e_\chi.$$

3.4.2. *A reformulation of the equivariant Tamagawa number conjecture.* Recall that

$$S = \{v_0, v_1, \ldots, v_n\}.$$

We start with the following lemma due to Burns.

Lemma 3.3. *There exists a finitely generated free $\mathbb{Z}[G]$-module F of rank d with $d \geq n$, a surjective morphism of $\mathbb{Z}[G]$-modules $\pi : F \twoheadrightarrow X_{K,S}$ and a $\mathbb{Z}[G]$-basis $\{b_1, \ldots, b_d\}$ of F such that*

(1) *$\pi(b_i) = w_i - w_0$, if $1 \leq i \leq n$,*
(2) *$\pi(b_i) \in Y_{K,\{v_0\}}$, meaning $\pi(b_i)$ is supported only at primes lying above v_0, if $n < i \leq d$.*

Proof. This follows from Lemma 7.1 of [6] with $A = 1$, the trivial subgroup. □

Hence, there exists a $\mathbb{Z}[G]$-projective resolution of $X_{K,S}$ of the form

$$P^\bullet : \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow F \xrightarrow{\pi} X_{K,S} \longrightarrow 0.$$

From now on, we suppose that all the computations relative to $\mathrm{Ext}^m_{\mathbb{Z}[G]}$ and $\mathrm{YExt}^m_{\mathbb{Z}[G]}$ of §2.1 are done with this particular choice of a $\mathbb{Z}[G]$-projective resolution of $X_{K,S}$.

We now introduce a finite set of primes T of k satisfying the following hypothesis:

Hypothesis 3.4. *One has*

(1) *$S \cap T = \varnothing$,*
(2) *$E_{K,S,T}$ has no $\mathbb{Z}$-torsion.*

Under this assumption, $E_{K,S,T}$ is a free $\mathbb{Z}$-module of $\mathbb{Z}$-rank $|S_K| - 1$. This will happen for instance if $T = \{\mathfrak{p}\}$ is a singleton and $(\mathfrak{p}, w_K) = 1$, where w_K is the number of roots of unity in K, since under this hypothesis, the roots of unity of K are distinct modulo $\mathfrak{P}$, where $\mathfrak{P}$ is any prime of K dividing $\mathfrak{p}$. For the rest of this section, we also make the following hypothesis.

Hypothesis 3.5. *The (S,T)-class group is trivial: $Cl_{K,S,T} = 1$.*

Under Hypothesis 3.5, one can also obtain a canonical class $c_{K/k,S,T} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S,T})$ as follows. Since $Cl_{K,S,T} = 1$, the exact sequence (2) just becomes

$$0 \longrightarrow E_{K,S,T} \longrightarrow E_{K,S} \longrightarrow \mathbb{F}_T^\times \longrightarrow 0. \tag{14}$$

Now, by Lemma 2.3, we have

$$H^m(G, \mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, \mathbb{F}_T^\times)) \xrightarrow{\simeq} \mathrm{Ext}^m_{\mathbb{Z}[G]}(X_{K,S}, \mathbb{F}_T^\times),$$

for all integers $m \geq 0$. Since $\mathbb{F}_T^\times$ is cohomologically trivial by Proposition 1.1, so is the $\mathbb{Z}[G]$-module $\mathrm{Hom}_{\mathbb{Z}}(X_{K,S}, \mathbb{F}_T^\times)$ and we conclude that

$$\mathrm{Ext}^m_{\mathbb{Z}[G]}(X_{K,S}, \mathbb{F}_T^\times) = 0,$$

for all $m \geq 1$. Applying the left exact functor $\mathrm{Hom}_{\mathbb{Z}[G]}(X_{K,S}, \_)$ to the short exact sequence (14), we obtain an isomorphism

$$\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S,T}) \xrightarrow{\simeq} \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S}).$$

The inverse image of the fundamental class $c_{K/k,S}$ under this isomorphism will be denoted by $c_{K/k,S,T}$.

Using the interpretation of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S,T})$ as the set of equivalence classes of 2-extensions of $X_{K,S}$ by $E_{K,S,T}$ recalled in §2.1, we obtain an exact sequence

$$0 \longrightarrow E_{K,S,T} \xrightarrow{\nu} \Omega_{S,T} \xrightarrow{\mu} F \xrightarrow{\pi} X_{K,S} \longrightarrow 0, \tag{15}$$

where $\Omega_{S,T}$ is a finitely generated $\mathbb{Z}[G]$-module. By definition of $c_{K/k,S,T}$, the push-out of (15) along the natural inclusion $E_{K,S,T} \hookrightarrow E_{K,S}$ gives a 2-extension of $X_{K,S}$ by $E_{K,S}$ representing $c_{K/k,S} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S})$. Indeed, this follows from the description of the map (4) recalled in §2.1. We then have the following diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & E_{K,S,T} & \xrightarrow{\nu} & \Omega_{S,T} & \xrightarrow{\mu} & F & \xrightarrow{\pi} & X_{K,S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\kappa} & & \| & & \| & & \\
0 & \longrightarrow & E_{K,S} & \xrightarrow{\nu'} & \Omega_S & \xrightarrow{\mu'} & F & \xrightarrow{\pi} & X_{K,S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \mathbb{F}_T^\times & & \mathrm{Coker}(\kappa) & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array} \tag{16}$$

where the first row represents $c_{K/k,S,T} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S,T})$, the second row represents $c_{K/k,S} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{K,S}, E_{K,S})$, the columns are exact, and Ω_S is the push-out of the diagram

$$\begin{array}{ccc}
E_{K,S,T} & \xrightarrow{\nu} & \Omega_{S,T} \\
\downarrow & & \\
E_{K,S} & &
\end{array}$$

The snake lemma implies that $\mathbb{F}_T^\times \simeq \mathrm{Coker}(\kappa)$ as $\mathbb{Z}[G]$-modules.

Since F is $\mathbb{Z}[G]$-free, Lemma 2.4 implies that Ω_S is cohomologically trivial; hence, the second row of (16) is a Tate sequence. From the second column of (16), and using the fact that $\mathbb{F}_T^\times$ is cohomologically trivial, we deduce that $\Omega_{S,T}$ is cohomologically trivial as well. But since $E_{K,S,T}$ is $\mathbb{Z}$-free, so is $\Omega_{S,T}$. It follows that $\Omega_{S,T}$ is $\mathbb{Z}[G]$-projective.

Starting from the first row of (16) and tensoring with $\mathbb{Q}$, we obtain an exact sequence of $\mathbb{Q}[G]$-modules

$$0 \longrightarrow \mathbb{Q}E_{K,S,T} \longrightarrow \mathbb{Q}\Omega_{S,T} \longrightarrow \mathbb{Q}F \longrightarrow \mathbb{Q}X_{K,S} \longrightarrow 0. \tag{17}$$

Since $E_{K,S,T}$ is of finite index in $E_{K,S}$, one has $\mathbb{Q}E_{K,S,T} \simeq \mathbb{Q}E_{K,S}$ as $\mathbb{Q}[G]$-modules. Moreover, $\mathbb{Q}E_{K,S} \simeq \mathbb{Q}X_{K,S}$ as $\mathbb{Q}[G]$-modules, because of the existence of an Artin system of units. Hence, $\mathbb{Q}E_{K,S,T} \simeq \mathbb{Q}X_{K,S}$ as $\mathbb{Q}[G]$-modules as well. Using the fact that $\mathbb{Q}[G]$ is semi-simple and breaking up (17) into two short exact sequences of $\mathbb{Q}[G]$-modules, one deduces the existence of an isomorphism of $\mathbb{Q}[G]$-modules between $\mathbb{Q}\Omega_{S,T}$ and $\mathbb{Q}F$.

Hence, by clearing up denominators if necessary, there exists an injective morphism of $\mathbb{Z}[G]$-modules

$$\psi : F \hookrightarrow \Omega_{S,T}.$$

Moreover, the index $(\Omega_{S,T} : \psi(F))$ is necessarily finite, because both F and $\Omega_{S,T}$ have the same $\mathbb{Z}$-rank.

Following Burns, we define $\phi \in \mathrm{End}_{\mathbb{Z}[G]}(F)$ to be the composition

$$\phi := \mu \circ \psi, \tag{18}$$

where $\mu : \Omega_{S,T} \longrightarrow F$ is the map showing up in the exact sequence (15). We then have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Ker}\phi & \longrightarrow & F & \xrightarrow{\phi} & F & \longrightarrow & \mathrm{Coker}\phi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \| & & \downarrow{\scriptstyle\tilde{\pi}} & & \\
0 & \longrightarrow & E_{K,S,T} & \xrightarrow{\nu} & \Omega_{S,T} & \xrightarrow{\mu} & F & \xrightarrow{\pi} & X_{K,S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\kappa} & & \| & & \| & & \\
0 & \longrightarrow & E_{K,S} & \xrightarrow{\nu'} & \Omega_S & \xrightarrow{\mu'} & F & \xrightarrow{\pi} & X_{K,S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \mathbb{F}_T^\times & \xrightarrow{\simeq} & \mathrm{Coker}(\kappa) & & & & & &
\end{array} \tag{19}$$

The map $\tilde{\pi}$ is the map induced by π, since $\mathrm{Im}(\phi) \subseteq \mathrm{Im}(\mu) = \mathrm{Ker}(\pi)$. Looking at the second column of (19), we get a short exact sequence of $\mathbb{Z}[G]$-modules of the form

$$0 \longrightarrow \Omega_{S,T}/\psi(F) \longrightarrow Q \longrightarrow \mathbb{F}_T^\times \longrightarrow 0, \tag{20}$$

where $Q = \Omega_S/\kappa \circ \psi(F)$ is a finite $\mathbb{Z}[G]$-module (the first map is induced by κ).

Lemma 3.6. *With the same notation as above, if* $((\Omega_{S,T} : \psi(F)), |G|) = 1$*, then* Q *is cohomologically trivial.*

Proof. Indeed, $\mathbb{F}_T^\times$ is cohomologically trivial by Proposition 1.1 and if the hypothesis

$$((\Omega_{S,T} : \psi(F)), |G|) = 1$$

is fulfilled, then

$$\widehat{H}^m(G', \Omega_{S,T}/\psi(F)) = 0,$$

for all $m \in \mathbb{Z}$ and all subgroups G' of G. One can then conclude the desired result by applying the long exact sequence in Tate cohomology to the short exact sequence (20). □

Lemma 3.7. *Let* ℓ *be a prime number relatively prime to* $(\Omega_{S,T} : \psi(F))$*. Then*

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Fit}_{\mathbb{Z}[G]}(Q) = \mathbb{Z}_\ell[G] \cdot \delta_T.$$

Proof. This is clear after applying the functor $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \_$ to the short exact sequence (20) and again using Proposition 1.1. □

From (19), we obtain the following short exact sequence of cochain complexes:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
C_\phi^\bullet : \dots & \longrightarrow & 0 & \longrightarrow & F & \xrightarrow{\phi} & F & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & \downarrow{\scriptstyle \kappa\circ\psi} & & \| & & & & \\
\Delta^\bullet : \dots & \longrightarrow & 0 & \longrightarrow & \Omega_S & \xrightarrow{\mu'} & F & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & \downarrow & & \downarrow & & & & \\
Q[0] : \dots & \longrightarrow & 0 & \longrightarrow & Q & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
\tag{21}
$$

where F, Ω_S, and Q are placed in degree 0. More succinctly, (21) can be written as

$$
0 \longrightarrow C_\phi^\bullet \longrightarrow \Delta^\bullet \longrightarrow Q[0] \longrightarrow 0. \tag{22}
$$

Lemma 3.8. *If* $((\Omega_{S,T} : \psi(F)), |G|) = 1$*, then* $Q[0]$ *is an object in* $\mathcal{C}^{perf}(\mathbb{Z}[G])$*. Moreover,* $\mathrm{Fit}_{\mathbb{Z}[G]}(Q)$ *is an invertible fractional ideal and*

$$
\mathrm{Det}_{\mathbb{Z}[G]}(Q[0]) = (\mathrm{Fit}_{\mathbb{Z}[G]}(Q)^{-1}, 0).
$$

Proof. Since Q is finite, there exists a positive integer t and a surjective morphism of $\mathbb{Z}[G]$-modules $f : \mathbb{Z}[G]^t \twoheadrightarrow Q$ which induces a short exact sequence of $\mathbb{Z}[G]$-modules

$$
0 \longrightarrow M \longrightarrow \mathbb{Z}[G]^t \xrightarrow{f} Q \longrightarrow 0,
$$

where M is the kernel of f. The $\mathbb{Z}[G]$-module M is $\mathbb{Z}$-free, since it is isomorphic to a submodule of $\mathbb{Z}[G]^t$. Lemma 3.6 implies that M is cohomologically trivial; therefore, it is $\mathbb{Z}[G]$-projective. Since $\mathbb{Z}[G]$ is a noetherian ring, we conclude that M is a finitely generated projective $\mathbb{Z}[G]$-module. We then have the following commutative diagram

$$
\begin{array}{ccccccccccc}
B^\bullet : \dots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}[G]^t & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\
Q[0] : \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Q & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}
$$

It is simple to check that this previous morphism of cochain complexes $B^\bullet \longrightarrow Q[0]$ is a quasi-isomorphism, that is $Q[0]$ is an object in $\mathcal{C}^{perf}(\mathbb{Z}[G])$. Hence, by definition of the $\mathrm{Det}_{\mathbb{Z}[G]}$ functor, we obtain

$$
\mathrm{Det}_{\mathbb{Z}[G]}(Q[0]) \simeq \mathrm{Det}_{\mathbb{Z}[G]}(M)^{-1} \otimes \mathrm{Det}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^t).
$$

Since Q has trivial rank, we see that M has constant rank t. Therefore, we obtain an isomorphism of $\mathbb{Z}[G]$-modules

$$
\left(\bigwedge_{\mathbb{Z}[G]}^{t} M \right) \otimes_{\mathbb{Z}[G]} \left(\bigwedge_{\mathbb{Z}[G]}^{t} (\mathbb{Z}[G]^t)^* \right) \xrightarrow{\simeq} \mathrm{Fit}_{\mathbb{Z}[G]}(Q),
$$

given by

$$
(m_1 \wedge \ldots \wedge m_t) \otimes (\varphi_1 \wedge \ldots \wedge \varphi_t) \mapsto \det(\varphi_i(m_j)).
$$

Using the fact that for a finitely generated projective $\mathbb{Z}[G]$-module M, one has $(M^*)^* \simeq M$, we conclude that

$$
\mathrm{Det}_{\mathbb{Z}[G]}(Q[0]) = (\mathrm{Fit}_{\mathbb{Z}[G]}(Q)^{-1}, 0),
$$

as we wanted to show. ∎

From the short exact sequence (22) and Lemma 3.8, we obtain

$$\mathrm{Det}_{\mathbb{Z}[G]}(\Delta^{\bullet}) \simeq \mathrm{Det}_{\mathbb{Z}[G]}(C_{\phi}^{\bullet}) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Q[0]) \simeq \mathrm{Det}_{\mathbb{Z}[G]}(C_{\phi}^{\bullet}) \otimes (\mathrm{Fit}_{\mathbb{Z}[G]}(Q)^{-1}, 0).$$

The equivariant Tamagawa number conjecture predicts that

$$\vartheta_{\Delta^{\bullet},\lambda}(\mathrm{Det}_{\mathbb{Z}[G]}(\Delta^{\bullet})) = (\mathbb{Z}[G] \cdot \theta^{*}_{K/k,S}(0), 0).$$

Letting $\lambda' = \tilde{\pi}^{-1} \circ \lambda \circ \psi$, a simple computation shows that

$$(\mathbb{Z}[G] \cdot \theta^{*}_{K/k,S}(0), 0) = \vartheta_{C^{\bullet}_{\phi},\lambda'}(\mathrm{Det}_{\mathbb{Z}[G]}(C_{\phi}^{\bullet})) \otimes (\mathrm{Fit}_{\mathbb{Z}[G]}(Q)^{-1}, 0), \tag{23}$$

where the isomorphism $\vartheta_{C^{\bullet}_{\phi},\lambda'} : \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}C^{\bullet}_{\phi}) \xrightarrow{\simeq} (\mathbb{C}[G], 0)$ is defined similarly as $\vartheta_{\Delta^{\bullet},\lambda}$.

It would be nice to know what $\vartheta_{C^{\bullet}_{\phi},\lambda'}(\mathrm{Det}_{\mathbb{Z}[G]}(C_{\phi}^{\bullet}))$ is and Burns proceeded as follows.

Starting with the two first rows of (19), we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathrm{Ker}\phi & \longrightarrow & F & \xrightarrow{\phi} & F & \longrightarrow & \mathrm{Coker}\phi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \| & & \downarrow{\scriptstyle\tilde{\pi}} & & \\
0 & \longrightarrow & E_{K,S,T} & \xrightarrow{\nu} & \Omega_{S,T} & \xrightarrow{\mu} & F & \xrightarrow{\pi} & X_{K,S} & \longrightarrow & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

After tensoring with $\mathbb{C}$, we get the following commutative diagram with exact rows whose columns are all isomorphisms of $\mathbb{C}[G]$-modules:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{C}\mathrm{Ker}\phi & \longrightarrow & \mathbb{C}F & \xrightarrow{\phi} & \mathbb{C}F & \longrightarrow & \mathbb{C}\mathrm{Coker}\phi & \longrightarrow & 0 \\
 & & \wr\!\downarrow{\scriptstyle\psi} & & \wr\!\downarrow{\scriptstyle\psi} & & \| & & \wr\!\downarrow{\scriptstyle\tilde{\pi}} & & \\
0 & \longrightarrow & \mathbb{C}E_{K,S,T} & \xrightarrow{\nu} & \mathbb{C}\Omega_{S,T} & \xrightarrow{\mu} & \mathbb{C}F & \xrightarrow{\pi} & \mathbb{C}X_{K,S} & \longrightarrow & 0
\end{array}$$

Consider the two short exact sequences:

$$0 \longrightarrow \mathbb{C}\mathrm{Ker}(\phi) \longrightarrow \mathbb{C}F \xrightarrow{\phi} \mathbb{C}\mathrm{Im}(\phi) \longrightarrow 0 \tag{24}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}\mathrm{Im}(\phi) & \longrightarrow & \mathbb{C}F & \longrightarrow & \mathbb{C}\mathrm{Coker}(\phi) & \longrightarrow & 0 \\
 & & & & & & \wr\!\downarrow{\scriptstyle\tilde{\pi}} & & \\
 & & & & & & \mathbb{C}X_{K,S} & &
\end{array} \tag{25}$$

Since $\mathbb{C}G$ is semi-simple, we can choose two sections

$$\iota_1 : \mathbb{C}\mathrm{Im}(\phi) \longrightarrow \mathbb{C}F$$

and

$$\iota_2 : \mathbb{C}X_{K,S} \longrightarrow \mathbb{C}F$$

of the short exact sequences (24) and (25) respectively.

Burns defines an endomorphism $f \in \mathrm{End}_{\mathbb{C}G}(\mathbb{C}F)$ as follows. Starting with the decomposition

$$\mathbb{C}F = \mathbb{C}\mathrm{Ker}(\phi) \oplus \iota_1(\mathbb{C}\mathrm{Im}(\phi)), \tag{26}$$

induced by the short exact sequence (24), one defines f to be equal to $\iota_2 \circ \lambda \circ \psi$ on $\mathbb{C}\mathrm{Ker}(\phi)$ and ϕ on $\iota_1(\mathbb{C}\mathrm{Im}(\phi))$. The morphism f can be pictured as follows:

$$\begin{array}{cccc} V = & \mathbb{C}\mathrm{Ker}(\phi) & \oplus & \iota_1(\mathbb{C}\mathrm{Im}(\phi)) \\ f\big\downarrow & \iota_2\circ\lambda\circ\psi\big\downarrow & & \phi\big\downarrow \\ V = & \iota_2(\mathbb{C}X_{K,S}) & \oplus & \mathbb{C}\mathrm{Im}(\phi) \end{array}$$

Since f is invertible, it makes sense to talk about $\det(f)$.

Theorem 3.9 (Burns)**.** *One has*

$$\vartheta_{C^\bullet_\phi,\lambda'}(\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_\phi)) = (\det(f) \cdot \mathbb{Z}[G], 0).$$

Proof. The $\mathbb{Z}[G]$-module $\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_\phi)$ is a free $\mathbb{Z}[G]$-module of rank one with a canonical generator given by

$$\omega = b_1 \wedge \ldots \wedge b_d \otimes b_1^* \wedge \ldots \wedge b_d^*.$$

It suffices to show that $\vartheta_{C^\bullet_\phi,\lambda'}(\omega) = \det(f)$ and this can be done one character at the time. Indeed, starting with

$$\vartheta_{C^\bullet_\phi,\lambda'} : \mathrm{Det}_{\mathbb{C}[G]}(\mathbb{C}C^\bullet_\phi) \xrightarrow{\simeq} (\mathbb{C}[G], 0),$$

and using the fact that the Det functor commutes with extension of scalars, we obtain maps

$$(\vartheta_{C^\bullet_\phi,\lambda'})^\chi : \mathrm{Det}_{\mathbb{C}[G]e_\chi}\left((\mathbb{C}C^\bullet_\phi)^\chi\right) \xrightarrow{\simeq} (\mathbb{C}[G]e_\chi, 0)$$

for each $\chi \in \widehat{G}$. Thus, it boils down to a statement about vector spaces which will be given in §3.5 below. □

Since the endomorphism f is invertible, one can define an element $x_T \in \mathbb{C}[G]^\times$ via the equality

$$\theta^*_{K/k,S,T}(0) = x_T \cdot \det(f).$$

Let us remark that x_T (as well as f) depends on the choice of ψ, ι_1, and ι_2. Combining the previous computations together, we get:

Proposition 3.10 (Burns)**.** *If $((\Omega_{S,T} : \psi(F)), |G|) = 1$, then the equivariant Tamagawa number conjecture is equivalent to the equality in $\mathbb{C}[G]$:*

$$\mathbb{Z}[G] \cdot x_T = \delta_T \cdot \mathrm{Fit}_{\mathbb{Z}[G]}(Q)^{-1}.$$

Proof. This follows from (23) and Theorem 3.9. □

Corollary 3.11. *Assuming $((\Omega_{S,T} : \psi(F)), |G|) = 1$, if the equivariant Tamagawa number conjecture is true, then*

1. *(1) $x_T \in \mathbb{Q}[G]$,*
2. *(2) $x_T \in \mathbb{Z}_\ell[G]$, for all $\ell \nmid (\Omega_{S,T} : \psi(F))$.*

Proof. This is clear from Proposition 3.10 and Lemma 3.7. □

Remark. Again, both f and x_T depend on the choice of ψ, ι_1, and ι_2. This is important, because in §6, we will apply Corollary 3.11 with different choices of ι_1, ι_2, and ψ.

3.5. **Proof of Theorem 3.9.** In this subsection, we complete the proof of Theorem 3.9. Let V be a finite dimensional D-vector space of dimension d and let $\phi \in \mathrm{End}_D(V)$. We have an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow V \xrightarrow{\phi} V \longrightarrow \mathrm{Coker}(\phi) \longrightarrow 0,$$

which can be broken into two short exact sequences

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow V \xrightarrow{\phi} \mathrm{Im}(\phi) \longrightarrow 0 \tag{27}$$

and

$$0 \longrightarrow \mathrm{Im}(\phi) \longrightarrow V \xrightarrow{p} \mathrm{Coker}(\phi) \longrightarrow 0. \tag{28}$$

Suppose we are given two sections

$$\iota_1 : \mathrm{Im}(\phi) \longrightarrow V$$

and

$$\iota_2 : \mathrm{Coker}(\phi) \longrightarrow V$$

of (27) and (28) respectively. Assume also that we are given an isomorphism of D-vector spaces

$$\lambda : \mathrm{Ker}(\phi) \xrightarrow{\simeq} \mathrm{Coker}(\phi).$$

Consider the isomorphism $\theta_\lambda : \mathrm{Det}_D(V) \otimes \mathrm{Det}_D(V)^{-1} \xrightarrow{\simeq} (D, 0)$ given by the composition of the following isomorphisms

$$\begin{aligned}
\mathrm{Det}_D(V) \otimes \mathrm{Det}_D(V)^{-1} &\xrightarrow{\simeq} (\mathrm{Det}_D(\mathrm{Ker}(\phi)) \otimes \mathrm{Det}_D(\mathrm{Im}(\phi))) \otimes (\mathrm{Det}_D(\mathrm{Im}(\phi)) \otimes \mathrm{Det}_D(\mathrm{Coker}(\phi)))^{-1} \\
&\xrightarrow{\simeq} \mathrm{Det}_D(\mathrm{Ker}(\phi)) \otimes \mathrm{Det}_D(\mathrm{Coker}(\phi))^{-1} \\
&\xrightarrow{\simeq} \mathrm{Det}_D(\mathrm{Coker}(\phi)) \otimes \mathrm{Det}_D(\mathrm{Coker}(\phi))^{-1} \\
&\overset{ev}{\xrightarrow{\simeq}} (D, 0),
\end{aligned}$$

where the first arrow comes from (27) and (28), the second from the evaluation map

$$\mathrm{Det}_D(\mathrm{Im}(\phi)) \otimes \mathrm{Det}_D(\mathrm{Im}(\phi))^{-1} \overset{ev}{\xrightarrow{\simeq}} (D, 0),$$

and the third one is $\mathrm{Det}_D(\lambda) \otimes id$.

The D-vector space $\mathrm{Det}_D(V) \otimes \mathrm{Det}_D(V)^{-1}$ is 1-dimensional with a canonical basis ω. This canonical basis ω is obtained as follows. Given any D-basis $(b_1, \ldots, b_d)$ of V, one has

$$\omega = b_1 \wedge \ldots \wedge b_d \otimes b_1^* \wedge \ldots \wedge b_d^*.$$

From (27), one has

$$V = \mathrm{Ker}(\phi) \oplus \iota_1(\mathrm{Im}(\phi))$$

and from (28)

$$V = \iota_2(\mathrm{Coker}(\phi)) \oplus \mathrm{Im}(\phi).$$

Let $f \in \mathrm{End}_D(V)$ be defined by the following diagram:

$$\begin{array}{ccccc}
V = & \mathrm{Ker}(\phi) & \oplus & \iota_1(\mathrm{Im}(\phi)) \\
{\scriptstyle f}\big\downarrow & {\scriptstyle \iota_2 \circ \lambda}\big\downarrow & & {\scriptstyle \phi}\big\downarrow \\
V = & \iota_2(\mathrm{Coker}(\phi)) & \oplus & \mathrm{Im}(\phi)
\end{array}$$

Proposition 3.12. *With the notation as above, one has*

$$\theta_\lambda(\omega) = \det(f).$$

Proof. Let $r = \dim_D(\mathrm{Ker}(\phi))$ so that $d - r = \dim_D(\mathrm{Im}(\phi))$. Let $(b_{r+1}, \dots, b_d)$ be a D-basis of $\mathrm{Im}(\phi)$ and let $(b_1, \dots, b_r)$ be a D-basis of $\mathrm{Ker}(\phi)$ so that

$$(b_1, \dots, b_r, \iota_1(b_{r+1}), \dots, \iota_1(b_d))$$

is a D-basis of V. Choose also $(b'_1, \dots, b'_r)$ a D-basis of $\mathrm{Coker}(\phi)$ such that

$$(\iota_2(b'_1), \dots, \iota_2(b_{r'}), b_{r+1}, \dots, b_d)$$

is a D-basis of V. Then, consider the following diagram:

$$\begin{array}{ccccc}
V' = & \mathrm{Ker}(\phi) & \oplus & \mathrm{Im}(\phi) \\
{\scriptstyle f'}\downarrow & {\scriptstyle id}\downarrow & & {\scriptstyle \iota_1}\downarrow \\
V = & \mathrm{Ker}(\phi) & \oplus & \iota_1(\mathrm{Im}(\phi)) \\
{\scriptstyle f}\downarrow & {\scriptstyle \iota_2\circ\lambda}\downarrow & & {\scriptstyle \phi}\downarrow \\
V = & \iota_2(\mathrm{Coker}(\phi)) & \oplus & \mathrm{Im}(\phi) \\
{\scriptstyle f''}\downarrow & {\scriptstyle p}\downarrow & & {\scriptstyle id}\downarrow \\
V'' = & \mathrm{Coker}(\phi) & \oplus & \mathrm{Im}(\phi)
\end{array}$$

We remark that $(b_1, \dots, b_d)$ is a D-basis of V' and $(b'_1, \dots, b'_r, b_{r+1}, \dots, b_d)$ is a D-basis of V''. With these choices of bases, we have

$$\det(f') = 1 = \det(f'').$$

Thus

$$\det(f) = \det(f'' \circ f \circ f'),$$

and $f'' \circ f \circ f'$ is given by the following diagram:

$$\begin{array}{ccccc}
V' = & \mathrm{Ker}(\phi) & \oplus & \mathrm{Im}(\phi) \\
{\scriptstyle f''\circ f\circ f'}\downarrow & {\scriptstyle \lambda}\downarrow & & {\scriptstyle id}\downarrow \\
V'' = & \mathrm{Coker}(\phi) & \oplus & \mathrm{Im}(\phi)
\end{array}$$

For $s = 1, \dots, r$, write

$$\lambda(b_s) = \sum_{t=1}^{r} a_{st} b'_t,$$

then

$$\det(f) = \det(a_{ij}). \tag{29}$$

Now, we have

$$\begin{aligned}
\omega &= b_1 \wedge \ldots \wedge b_r \wedge \iota_1(b_{r+1}) \wedge \ldots \wedge \iota_1(b_d) \otimes b_1^* \wedge \ldots \wedge b_r^* \wedge \iota_1(b_{r+1})^* \wedge \ldots \wedge \iota_1(b_d)^* \\
&= \iota_2(b'_1) \wedge \ldots \wedge \iota_2(b'_r) \wedge b_{r+1} \wedge \ldots \wedge b_d \otimes \iota_2(b'_1)^* \wedge \ldots \wedge \iota_2(b'_r)^* \wedge b_{r+1}^* \wedge \ldots \wedge b_d^*.
\end{aligned}$$

If we follow the arrows in the definition of the isomorphism θ_λ, we obtain

$$\begin{aligned}
\omega &\mapsto (b_1 \wedge \ldots \wedge b_r \otimes b_{r+1} \wedge \ldots \wedge b_d) \otimes \left(b_{r+1}^* \wedge \ldots \wedge b_d^* \otimes b_1'^* \wedge \ldots \wedge b_r'^*\right) \\
&\mapsto b_1 \wedge \ldots \wedge b_r \otimes b_1'^* \wedge \ldots \wedge b_r'^* \\
&\mapsto \lambda(b_1) \wedge \ldots \wedge \lambda(b_r) \otimes b_1'^* \wedge \ldots \wedge b_r'^* \\
&\mapsto \det(b_i'^*(\lambda(b_j))).
\end{aligned}$$

A simple computation shows that $b_i'^*(\lambda(b_j)) = a_{ji}$; therefore,

$$\det(b_i'^*(\lambda(b_j))) = \det(a_{ij}) = \det(f),$$

by (29). □

4. The extended abelian Stark conjecture and a stronger conjecture

In this section, we recall the statement of the Emmons-Popescu conjecture. Moreover, we state a stronger conjecture, namely Conjecture 4.16, which is the main object of study of this paper. In [28] and [27], we introduced a Question which we now also generalize to the higher order of vanishing situation. Moreover, we study the relationship between Conjecture 4.16 and this Question and we show that if one assumes the equivariant Tamagawa number conjecture, or solely the Rubin-Stark conjecture, then they are equivalent.

4.1. Basic definitions.

In the formulation of the classical abelian rank r Stark conjecture (the Rubin-Stark conjecture), one of the hypotheses is that the set S should contain at least r places which split completely. This is to guarantee that

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) \geq r,$$

for all non-trivial characters $\chi \in \widehat{G}$, because of the following theorem:

Theorem 4.1. *Let K/k be an abelian extension of number fields and S a finite set of primes of k containing the infinite ones. We have*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \dim_{\mathbb{C}}(\mathbb{C}E_{K,S} \cdot e_\chi) = \begin{cases} |S|-1, & \text{if } \chi = \chi_1, \\ |\{v \in S \,|\, G_v \subseteq \mathrm{Ker}(\chi)\}|, & \text{if } \chi \neq \chi_1. \end{cases}$$

Proof. See [26], page 24, Proposition 3.4. □

The purpose of the extended abelian Stark conjecture is to remove this hypothesis and replace it with the more general notion of r-cover:

Definition 4.2. Let K/k be an abelian extension of number fields with Galois group G and let Λ be any subset of $\widehat{G}$. Let S be any finite set of primes of k (not necessarily containing the ramified nor the archimedean primes). Let $r \geq 1$ be an integer. The set S is said to be an *r-cover* for Λ if the following two conditions hold:

1. For all non-trivial characters $\chi \in \Lambda$, there exist at least r places $v \in S$ such that $G_v \subseteq \mathrm{Ker}(\chi)$,
2. If the trivial character is in Λ, then $|S| \geq r+1$.

In the case where $\Lambda = \widehat{G}$, we also say that S is an r-cover for G (or for K/k), rather than for $\widehat{G}$.

If S is an r-cover for K/k containing $S(K/k)$, then $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) \geq r$ for all $\chi \in \widehat{G}$ by Theorem 4.1.

Unless otherwise stated, we always suppose that an r-cover S for K/k contains $S(K/k)$.

Remark. Let S be an r-cover. Lemma 2.2 of [11] shows that if $|S| = r+1$, then S has to contain r places which split completely; thus, we are in the classical setting of the Rubin-Stark conjecture. Burns showed in [6] that the equivariant Tamagawa number conjecture implies the Rubin-Stark conjecture. Therefore, we can avoid to talk about this case and we will typically assume that $|S| \geq r+2$ if S is an r-cover. This implies that $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi_1) \geq r+1$, where χ_1 is the trivial character of G.

Definition 4.3. Let K/k be an abelian extension of number fields with Galois group G and S an r-cover for K/k. The set of characters $\chi \in \widehat{G}$ whose S-imprimitive L-functions have order of vanishing precisely r will be denoted by $\widehat{G}_{r,S}$. One also defines the idempotent

$$e_{r,S} = \sum_{\chi \in \widehat{G}_{r,S}} e_\chi.$$

The idempotent $e_{r,S}$ is in fact a rational idempotent, since $\widehat{G}_{r,S}$ is closed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by Theorem 4.1.

Definition 4.4. If S is an r-cover for K/k, one defines S_{min} as follows: it consists of all primes $v \in S$ for which there exists $\chi \in \widehat{G}_{r,S}$, $\chi \neq \chi_1$, such that $G_v \subseteq \mathrm{Ker}(\chi)$.

In other words, S_{min} precisely consists of the places in S which are responsible for the vanishing of the S-imprimitive L-functions associated to non-trivial characters having order of vanishing exactly r. We note that if $S_{min} \neq \varnothing$, then $|S_{min}| \geq r$ necessarily.

We now let T be another finite set of finite primes satisfying Hypothesis 3.4.

4.2. **The rank one case.** Let S be a finite set of primes containing $S(K/k)$ and suppose it is a 1-cover satisfying $S \neq S_{min}$ and $|S| \geq 3$. We also suppose that $S_{min} \neq \varnothing$ otherwise the extended abelian rank one Stark conjecture is meaningless. We write $S = \{v_0, v_1, \ldots, v_n\}$, so that

$$S_{min} = \{v_1, \ldots, v_m\},$$

for some m satisfying $1 \leq m \leq n$.

We partition the set $\widehat{G}_{1,S}$ as follows:

$$\widehat{G}_{1,S} = \bigcup_{i=1}^{m} \widehat{G}_{1,S,i},$$

where for $i = 1, \ldots, m$, we define

$$\widehat{G}_{1,S,i} = \{\chi \in \widehat{G}_{1,S} \,|\, G_i \subseteq \mathrm{Ker}(\chi)\}.$$

This union is disjoint, because of the definition of S_{min}: if $\chi \in \widehat{G}_{1,S}$, then there is a unique $i \in \{1, \ldots, m\}$ such that $G_i \subseteq \mathrm{Ker}(\chi)$.

Furthermore, if we set

$$e_i = e_{1,S,v_i} := \sum_{\chi \in \widehat{G}_{1,S,i}} e_\chi,$$

then

$$e_{1,S} = \sum_{i=1}^{m} e_i.$$

Moreover, the e_i are also rational idempotents for $i = 1, \ldots, m$, since each of the sets $\widehat{G}_{1,S,i}$ is closed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proposition 3.2 of [11] shows that the map

$$R_{K/k,S} : \mathbb{C}E_{K,S,T} \cdot e_{1,S} \longrightarrow \mathbb{C}[G] \cdot e_{1,S},$$

defined on units $u \in E_{K,S,T}$ by the formula

$$R_{K/k,S}(u) = \sum_{i=1}^{m} \ell_i(u)$$

is an isomorphism of $\mathbb{C}[G]$-modules.

The S-version of the following conjecture is due to Erickson-Stark, (Conjecture 4.1 of [12]), whereas the (S,T)-version presented here is the rank one case of the Emmons-Popescu conjecture.

Conjecture 4.5 (Emmons-Popescu). *Let K/k be a finite abelian extension of number fields and S a 1-cover satisfying $|S| \geq 3$. Suppose also that $S \neq S_{min}$ and let T be another finite set of primes of k satisfying Hypothesis 3.4. Let $\eta \in \mathbb{C}E_{K,S,T} \cdot e_{1,S}$ be the unique element such that*

$$R_{K/k,S}(\eta) = e_{1,S} \cdot \theta^*_{K/k,S,T}(0).$$

Then $\eta \in E_{K,S,T}$.

If we restrict $R_{K/k,S}$ further to $\mathbb{C}E_{K,S,T} \cdot e_i$ for $i = 1, \ldots, m$, we obtain an isomorphism of $\mathbb{C}[G]$-modules

$$R_{K/k,S} : \mathbb{C}E_{K,S,T} \cdot e_i \longrightarrow \mathbb{C}[G] \cdot e_i.$$

Lemma 4.6. *Let v be a place of k and $\chi \in \widehat{G}$ such that $\chi \neq \chi_1$. Fix a place w of K lying above v. If $G_v \not\subseteq \mathrm{Ker}(\chi)$, then*

$$\ell_w(u \cdot e_\chi) = 0,$$

for all $u \in E_{K,S,T}$. In other words, ℓ_w is trivial on $\mathbb{C}E_{K,S,T} \cdot e_\chi$.

Proof. Fix $\{\sigma_1, \ldots, \sigma_s\}$, a complete set of representatives of G/G_v. Then

$$\begin{aligned}\ell_w(u \cdot e_\chi) &= e_\chi \cdot \ell_w(u) \\ &= -e_\chi \cdot \frac{1}{|G_v|} \sum_{\sigma \in G} \log|u^\sigma|_w \cdot \sigma^{-1} \\ &= -e_\chi \cdot \frac{1}{|G_v|} \sum_{t=1}^{s} \sum_{h \in G_v} \log|u^{\sigma_t h}|_w (\sigma_t h)^{-1} \\ &= -e_\chi \cdot N_v \cdot \frac{1}{|G_v|} \sum_{t=1}^{s} \log|u^{\sigma_t}|_w \cdot \sigma_t^{-1},\end{aligned}$$

where

$$N_v = \sum_{h \in G_v} h.$$

Since $G_v \not\subseteq \mathrm{Ker}(\chi)$, we have $e_\chi \cdot N_v = 0$. □

From Lemma 4.6, we conclude that on the space $\mathbb{C}E_{K,S,T} \cdot e_i$, one has $R_{K/k,S} = \ell_i$. We now introduce a stronger conjecture.

Conjecture 4.7. *Let K/k be a finite abelian extension of number fields and S a 1-cover such that $|S| \geq 3$. Suppose also that $S \neq S_{min}$ and let T be another finite set of primes of k satisfying Hypothesis 3.4. For $i = 1, \ldots, m$, let $\eta_i \in \mathbb{C}E_{K,S,T} \cdot e_i$ be the unique element such that*

$$\ell_i(\eta_i) = e_i \cdot \theta^*_{K/k,S,T}(0).$$

Then $\eta_i \in E_{K,S,T}$.

Lemma 4.8. *Conjecture 4.7 implies Conjecture 4.5.*

Proof. This is clear, since

$$\eta = \sum_{i=1}^{m} \eta_i.$$

□

Conjecture 4.7 and its generalization to higher order of vanishing situations (Conjecture 4.16 below) are the main object of study of this paper.

The rest of this section is devoted to the explanation of the relationship between Conjecture 4.7 and a Question introduced in [27] and [28]. In the sequel, we let $L_i = K^{G_i}$, $\Gamma_i = G/G_i$, and $n_i = |G_i|$, for each $i = 1, \ldots, m$. Let us remark that since $E_{K,S,T}$ is assumed to be without $\mathbb{Z}$-torsion, one has injections

$$E_{L_i,S,T} \hookrightarrow \mathbb{C}E_{L_i,S,T} \hookrightarrow \mathbb{C}E_{K,S,T},$$

for all $i = 1, \ldots, m$. Let $\pi_i : G \twoheadrightarrow \Gamma_i$ be the natural projection map. Because of the duality theory for finite abelian groups, we have a bijection

$$\widehat{\Gamma}_i \longleftrightarrow \{\chi \in \widehat{G} \,|\, G_i \subseteq \mathrm{Ker}(\chi)\},$$

given by $\chi \mapsto \chi \circ \pi_i$. From the inflation properties of L-functions this bijection induces the following bijection

$$(\widehat{\Gamma}_i)_{1,S} \longleftrightarrow \widehat{G}_{1,S,i}. \tag{30}$$

It follows that for L_i/k, S is still a 1-cover and $S_{min} = \{v_i\}$. Therefore, at the level of L_i/k, we are in the classical setting of the abelian rank one Stark conjecture, since v_i splits completely in L_i/k. For the extension L_i/k, we have the idempotent

$$e'_{1,S} = \sum_{\chi' \in (\widehat{\Gamma}_i)_{1,S}} e_{\chi'} \in \mathbb{C}[\Gamma_i],$$

where $e_{\chi'}$ is the idempotent in $\mathbb{C}[\Gamma_i]$ corresponding to χ'.

We need to introduce the restriction and corestriction maps. Given a tower of fields $k \subseteq M \subseteq K$, let H be the Galois group of K/M, G the one of K/k, and $\Gamma = G/H$ the one of M/k.

The restriction map

$$\mathrm{res}_{K/M} : \mathbb{C}[G] \longrightarrow \mathbb{C}[\Gamma]$$

is defined on $\sigma \in G$ by the formula

$$\sigma \mapsto \sigma|_M.$$

It is a $\mathbb{C}$-algebra morphism. One also has the usual corestriction map

$$\mathrm{cor}_{K/M} : \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[G]$$

defined by

$$\gamma \mapsto \sum_{\substack{\sigma \in G \\ \sigma|_M = \gamma}} \sigma = \tilde{\gamma} \cdot N_H,$$

where $\tilde{\gamma}$ is any extension of γ and

$$N_H = \sum_{h \in H} h.$$

The corestriction map is only a $\mathbb{C}[G]$-module morphism and it satisfies

$$\mathrm{cor}_{K/M}(\lambda_1 \cdot \lambda_2) = \frac{1}{|H|} \mathrm{cor}_{K/M}(\lambda_1) \cdot \mathrm{cor}_{K/M}(\lambda_2), \tag{31}$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}[\Gamma]$. One has

$$\mathrm{cor}_{K/M} \circ \mathrm{res}_{K/M}(\lambda) = N_H \cdot \lambda, \tag{32}$$

for all $\lambda \in \mathbb{C}[G]$. Also,

$$\mathrm{res}_{K/M} \circ \mathrm{cor}_{K/M}(\lambda) = |H| \cdot \lambda, \tag{33}$$

for all $\lambda \in \mathbb{C}[\Gamma]$.

From (30), it follows that

$$\mathrm{res}_{K/L_i}(e_i) = e'_{1,S}.$$

We will need the following lemma.

Lemma 4.9. *Let K/k be a finite abelian extension of number fields and fix a place $v \in S$. Let also w be a place of K lying above v. As above, let M be an intermediate field and let w' be the unique place of M lying between v and w. We let $\ell'_{w'}$ denotes the regulator map at the level of M/k. If $u \in E_{M,S}$, then*

$$\ell_w(u) = \mathrm{cor}_{K/M}(\ell'_{w'}(u)).$$

Proof. As above, let H be the Galois group of K/M and $\Gamma = G/H$ the one of M/k. Starting with $u \in E_{M,S}$ and a complete set of representatives $\{\sigma_1, \ldots, \sigma_s\}$ of Γ, we have

$$\begin{aligned}\ell_w(u) &= -\frac{1}{|G_v|}\sum_{\sigma\in G}\log|u^{\sigma}|_w \cdot \sigma^{-1}\\ &= -\frac{1}{|G_v|}\sum_{t=1}^{s}\sum_{h\in H}\log|u^{\sigma_t h}|_w(\sigma_t h)^{-1}\\ &= -\left(\frac{1}{|G_v|}\sum_{t=1}^{s}\log|u^{\sigma_t}|_w\sigma_t^{-1}\right)\cdot N_H\\ &= -\frac{|H_{w'}|}{|G_v|}\left(\sum_{t=1}^{s}\log|u^{\sigma_t}|_{w'}\sigma_t^{-1}\right)\cdot N_H,\end{aligned}$$

the last line being true, since $|u|_w = |u|_{w'}^{|H_{w'}|}$. We then conclude that

$$\ell_w(u) = -\frac{1}{|\Gamma_v|}\left(\sum_{t=1}^{s}\log|u^{\sigma_t}|_{w'}\sigma_t^{-1}\right)\cdot N_H = \mathrm{cor}_{K/M}(\ell'_{w'}(u)).$$

□

Coming back to the setting before the lemma, we let w_i' be the place of L_i lying between v_i and w_i. We also let ℓ_i' be the corresponding regulator map for the extension L_i/k, that is

$$\ell_i'(u) = -\sum_{\gamma\in\Gamma_i}\log|u^{\gamma}|_{w_i'}\cdot\gamma^{-1},$$

whenever $u \in E_{L_i,S,T}$. Let η_i' be the unique element of $\mathbb{C}E_{L_i,S,T}\cdot e'_{1,S}$ be such that

$$\ell_i'(\eta_i') = e'_{1,S}\cdot\theta^*_{L_i/k,S,T}(0).$$

The classical abelian rank one Stark conjecture predicts that $\eta_i' \in E_{L_i,S,T}$.

Proposition 4.10. *The notation being as above, one has the following equality in $\mathbb{C}E_{K,S,T}$:*

$$\eta_i = \frac{1}{n_i}\eta_i',$$

where we remind the reader that $n_i = |G_i|$.

Proof. By definition of η_i', we have

$$\eta_i' = e'_{1,S}\cdot\eta_i' = \mathrm{res}_{K/L_i}(e_i)\cdot\eta_i' = e_i\cdot\eta_i'.$$

Hence, we just have to show that

$$e_\chi\cdot\eta_i = e_\chi\cdot\frac{1}{n_i}\eta_i',$$

for all $\chi \in \widehat{G}_{1,S,i}$.

On one hand, we have

$$\begin{aligned}\ell_i(e_\chi\cdot\eta_i) &= e_\chi\cdot e_i\cdot\theta^*_{K/k,S,T}(0)\\ &= e_\chi\cdot\theta^*_{K/k,S,T}(0).\end{aligned}$$

On the other hand, it follows from Lemma 4.9 and (32) that if $\chi \in \widehat{G}_{1,S,i}$, then

$$\begin{aligned}
\ell_i\left(e_\chi \cdot \frac{1}{n_i}\eta_i'\right) &= e_\chi \cdot \frac{1}{n_i}\ell_i(\eta_i') \\
&= e_\chi \cdot \frac{1}{n_i}\mathrm{cor}_{K/L_i}(\ell_i'(\eta_i')) \\
&= e_\chi \cdot \frac{1}{n_i}\mathrm{cor}_{K/L_i}(e_{1,S}' \cdot \theta^*_{L_i/k,S,T}(0)) \\
&= e_\chi \cdot \frac{1}{n_i}\mathrm{cor}_{K/L_i}(\mathrm{res}_{K/L_i}(e_i \cdot \theta^*_{K/k,S,T}(0))) \\
&= e_\chi \cdot \frac{1}{n_i} \cdot N_i \cdot e_i \cdot \theta^*_{K/k,S,T}(0) \\
&= e_\chi \cdot \theta^*_{K/k,S,T}(0).
\end{aligned}$$

Thus,

$$\ell_i(e_\chi \cdot \eta_i) = \ell_i\left(e_\chi \cdot \frac{1}{n_i}\eta_i'\right),$$

for all $\chi \in \widehat{G}_{1,S,i}$ and we conclude that

$$\eta_i = \frac{1}{n_i}\eta_i',$$

since $\ell_i : \mathbb{C}E_{K,S,T} \cdot e_i \longrightarrow \mathbb{C}[G] \cdot e_i$ is an isomorphism of $\mathbb{C}[G]$-modules. □

Proposition 4.11. *With the same notation as before, let us suppose that the usual abelian rank one Stark conjecture for the data $(L_i/k, S, T, v_i)$ is true, that is $\eta_i' \in E_{L_i,S,T}$. If Conjecture 4.7 is valid, then*

$$\eta_i \in L_i^\times \cap E_{K,S,T} = E_{L_i,S,T}.$$

Proof. Assuming the abelian rank one Stark conjecture for L_i/k, we have $\eta_i' \in E_{L_i,S,T}$. The validity of Conjecture 4.7 implies that $\eta_i \in E_{K,S,T}$. In multiplicative notation now, Proposition 4.10 implies

$$\eta_i^{n_i} = \eta_i'.$$

We will be done if we can show that $\eta_i \in L_i$. If $\sigma \in G_i$, then $\eta_i^{\sigma-1}$ is an n_i-th root of unity, since $\eta_i^{n_i} = \eta_i'$ is fixed under G_i. On the other hand, $\eta_i^{\sigma-1}$ is in $E_{K,S,T}$ which has no torsion. So $\eta_i^{\sigma-1} = 1$, and η_i is fixed under G_i. Hence $\eta_i \in L_i$. □

In [28] and [27], we studied a Question for which one objective of this paper is to show that it has an affirmative answer when the base field is $\mathbb{Q}$. We will now explain that an affirmative answer to this Question is equivalent to Conjecture 4.7 under the assumption of the classical abelian rank one Stark conjecture. Let us first recall the statement of this Question which we now upgrade to the status of a Conjecture. We fix an integer i with $1 \le i \le m$, and as before, we let

$$e_{1,S}' = \sum_{\chi \in (\widehat{\Gamma}_i)_{1,S}} e_\chi.$$

Conjecture 4.12 ($St(K/k, S, T, v_i)$)**.** *Let K/k be a finite abelian extension of number fields and S a 1-cover. Suppose that $S \neq S_{min}$ and $|S| \geq 3$. Let T be another finite set of primes of k satisfying Hypothesis 3.4. Then, there exists $\varepsilon_i \in E_{L_i,S,T}$ satisfying $e_{1,S}' \cdot \varepsilon_i = \varepsilon_i$, and such that*

$$\theta'_{L_i/k,S,T}(0) = -n_i \sum_{\gamma \in \Gamma_i} \log |\varepsilon_i^\gamma|_{w_i'} \cdot \gamma^{-1},$$

where w_i' is the place of L_i lying between v_i and w_i.

Proposition 4.13. *Assuming the classical abelian rank one Stark conjecture, Conjectures 4.7 and 4.12 are equivalent.*

Proof. Let us assume first that Conjecture 4.7 is true. Then, it is a direct consequence of Propositions 4.10 and 4.11 that Conjecture 4.12 is true. One would then have $\varepsilon_i = \eta_i \in E_{L_i,S,T}$.

Conversely, let us assume that Conjecture 4.12 is true. Then, one has $\eta_i' = \varepsilon_i^{n_i}$, since

$$\ell_i' : \mathbb{C}E_{L_i,S,T} \cdot e_{1,S}' \longrightarrow \mathbb{C}[G] \cdot e_{1,S}'$$

is an isomorphism of $\mathbb{C}[G]$-modules. From Proposition 4.10, we obtain

$$\eta_i = \frac{1}{n_i}\eta_i' = \frac{1}{n_i}\varepsilon_i^{n_i} = \varepsilon_i \in E_{L_i,S,T} \subseteq E_{K,S,T}.$$

□

Remark. Let us suppose that S is a finite set of places of k containing $S(K/k)$ satisfying $|S| \geq 3$. If S contains a place v which splits completely in K/k, then S is automatically a 1-cover. Moreover, either $S_{min} = \varnothing$ or $S_{min} = \{v\}$. In the former case, S is in fact a 2-cover and the extended abelian *rank one* Stark conjecture is void. In the latter case, Conjectures 4.5, 4.7, and 4.12 all reduce to the classical (S,T)-version of the abelian rank one Stark conjecture.

Let us end this section with another remark. In view of Propositions 4.10 and 4.11, one could change the prediction $\eta_i \in E_{K,S,T}$ of Conjecture 4.7 to $\eta_i \in E_{L_i,S,T}$. Conjecture 4.12 would then be a direct consequence of Conjecture 4.7 by applying the restriction map.

4.3. **The higher order of vanishing case.** In this section, we present the conjecture of Emmons-Popescu in the higher order of vanishing case. We also generalize Conjectures 4.7 and 4.12 to the higher order of vanishing situation.

We let S be a finite set of places of k containing $S(K/k)$. Moreover, we suppose that S is an r-cover satisfying $S \neq S_{min}$ and $|S| \geq r+2$. We also suppose that $S_{min} \neq \varnothing$ in which case $|S_{min}| \geq r$ necessarily. We write $S = \{v_0, v_1, \ldots, v_n\}$ so that

$$S_{min} = \{v_1, \ldots, v_m\},$$

for some m satisfying $r \leq m \leq n$.

As in the rank one case, we partition the set $\widehat{G}_{r,S}$ as follows. We let $\wp_r(m)$ denote the set of r-tuples $(n_1, \ldots, n_r)$ of integers between 1 and m satisfying

$$n_1 < \ldots < n_r.$$

Then, we have

$$\widehat{G}_{r,S} = \bigcup_{I \in \wp_r(m)} \widehat{G}_{r,S,I},$$

where for $I \in \wp_r(m)$, we define

$$\widehat{G}_{r,S,I} = \{\chi \in \widehat{G}_{r,S} \,|\, G_i \subseteq \mathrm{Ker}(\chi), \text{ for all } i \in I\}.$$

Again, the union is disjoint, because of the definition of S_{min}: if $\chi \in \widehat{G}_{r,S}$, then there is a unique $I \in \wp_r(m)$ such that $G_i \subseteq \mathrm{Ker}(\chi)$ for all $i \in I$.

As in the rank one case, we define new rational idempotents as follows: for $I \in \wp_r(m)$, we set

$$e_I = e_{r,S,I} := \sum_{\chi \in \widehat{G}_{r,S,I}} e_\chi.$$

Then,

$$e_{r,S} = \sum_{I \in \wp_r(m)} e_I.$$

Following [11], for $I \in \wp_r(m)$, we define a $\mathbb{C}[G]$-module morphism

$$R_I : \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \longrightarrow \mathbb{C}[G],$$

by the formula

$$R_I(u_1 \wedge u_2 \wedge \ldots \wedge u_r) = \det(\ell_i(u_j))_{i \in I, j=1,\ldots,r},$$

where $u_i \in E_{K,S,T}$ and then extending by $\mathbb{C}$-linearity. Again, Proposition 3.2 of [11] shows that the map

$$R_{K/k,S} : \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \cdot e_{r,S} \longrightarrow \mathbb{C}[G] \cdot e_{r,S},$$

defined by

$$R_{K/k,S}(u) = \sum_{I \in \wp_r(m)} R_I(u),$$

is an isomorphism of $\mathbb{C}[G]$-modules.

As before, Emmons and Popescu proceed and define

$$\eta = \eta_{K/k,S,T,r} \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \cdot e_{r,S}$$

to be the unique element such that

$$R_{K/k,S}(\eta) = e_{r,S} \cdot \theta^*_{K/k,S,T}(0).$$

It would then be tempting to conjecture that η lies in the image of $\wedge^r_{\mathbb{Z}[G]} E_{K,S,T}$ in $\mathbb{Q} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T}$ under the natural $\mathbb{Z}[G]$-module morphism

$$\bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \longrightarrow \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}, \tag{34}$$

but Rubin exhibited counter-examples in [23] already in the classical setting where there are r places in S which split completely in K/k. See also [21] and [22]. From now on, the image of the map (34) will be denoted by

$$\overline{\bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}}.$$

In the classical setting, allowing some denominators in a subtle way, Rubin defined a slightly bigger lattice than $\overline{\wedge^r_{\mathbb{Z}[G]} E_{K,S,T}}$ in $\mathbb{Q} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T}$, denoted by $\Lambda_{K/k,S,T}$, and conjectured that $\eta \in \Lambda_{K/k,S,T}$. Emmons and Popescu used the same lattice in the extended situation. The Rubin lattice is defined as follows:

$$\Lambda_{K/k,S,T} := \Big\{ u \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \;\Big|\; \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_r(u) \in \mathbb{Z}[G], \text{ for all } \varphi_i \in E^*_{K,S,T} \Big\}.$$

If $r = 1$, the Rubin lattice is nothing else than $E_{K,S,T}$ as the following lemma shows.

Lemma 4.14. *If $r = 1$, then*

$$\Lambda_{K/k,S,T} = E_{K,S,T}.$$

Proof. See Proposition 1.2 of [23]. □

Conjecture 4.15 (Emmons-Popescu)**.** *Let K/k be a finite abelian extension of number fields and S an r-cover satisfying $|S| \geq r+2$. Suppose also that $S \neq S_{min}$ and let T be another finite set of primes of k satisfying Hypothesis 3.4. Let $\eta \in \mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_{r,S}$ be the unique element such that*

$$R_{K/k,S}(\eta) = e_{r,S} \cdot \theta^*_{K/k,S,T}(0).$$

Then $\eta \in \Lambda_{K/k,S,T}$.

If we restrict $R_{K/k,S}$ further to $\mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I$, for $I \in \wp_r(m)$, we obtain an isomorphism of $\mathbb{C}[G]$-modules

$$R_{K/k,S} : \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \cdot e_I \longrightarrow \mathbb{C}[G] \cdot e_I.$$

Similarly, as in the rank one case, on the space $\mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I$, one has $R_{K/k,S} = R_I$. Indeed, this follows again from Lemma 4.6.

We now extend Conjecture 4.7 to the higher order of vanishing situation.

Conjecture 4.16. *Let K/k be a finite abelian extension of number fields and S an r-cover such that $|S| \geq r+2$. Suppose also that $S \neq S_{min}$ and let T be another finite set of primes of k satisfying Hypothesis 3.4. For $I \in \wp_r(m)$, let $\eta_I \in \mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I$ be the unique element such that*

$$R_I(\eta_I) = e_I \cdot \theta^*_{K/k,S,T}(0).$$

Then $\eta_I \in \Lambda_{K/k,S,T}$.

This last conjecture is the main object of study of this paper. We will see that it is implied by the equivariant Tamagawa number conjecture.

Lemma 4.17. *Conjecture 4.16 implies Conjecture 4.15.*

Proof. This is clear, since

$$\eta = \sum_{I \in \wp_r(m)} \eta_I.$$

□

We will now extend Conjecture 4.12 to the higher order of vanishing situation. For $I \in \wp_r(m)$, set $D_I = \langle G_i \,|\, i \in I \rangle$. In the sequel, we let $L_I = K^{D_I}$, $\Gamma_I = G/D_I$, and $n_I = |D_I|$. Let us remark that if $I, J \in \wp_r(m)$ and $I \neq J$, then $L_I \neq L_J$. Since $\mathbb{C}[G]$ is semi-simple, we have an injective $\mathbb{C}[G]$-module morphism

$$\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{L_I,S,T} \hookrightarrow \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}.$$

Moreover,

$$\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{L_I,S,T} \simeq \mathbb{C} \bigwedge_{\mathbb{Z}[\Gamma_I]}^{r} E_{L_I,S,T},$$

as $\mathbb{C}[G]$-modules; therefore, we obtain an injective $\mathbb{C}[G]$-module morphism

$$\mathbb{C} \bigwedge_{\mathbb{Z}[\Gamma_I]}^{r} E_{L_I,S,T} \hookrightarrow \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}.$$

As in the rank one situation, the projection map $\pi_I : G \twoheadrightarrow \Gamma_I$ induces a bijection

$$\widehat{\Gamma}_I \longleftrightarrow \{\chi \in \widehat{G} \,|\, D_I \subseteq \mathrm{Ker}(\chi)\},$$

which in turns induces the following bijection

$$(\widehat{\Gamma}_I)_{r,S} \longleftrightarrow \widehat{G}_{r,S,I}. \tag{35}$$

It follows that for L_I/k, the set S is still an r-cover and $S_{min} = \{v_i \,|\, i \in I\}$. The places v_i, for which $i \in I$, split completely in L_I/k; therefore, we are in the setting of the classical Rubin-Stark conjecture. As in the rank one case, if we let

$$e'_{r,S} = \sum_{\chi' \in (\widehat{\Gamma}_I)_{r,S}} e_{\chi'},$$

then it follows from (35) that

$$\mathrm{res}_{K/L_I}(e_I) = e'_{r,S}.$$

For each $i \in I$, we let w_i' be the unique place of L_I lying between v_i and w_i and we let R_I' be the corresponding regulator map at the level of L_I/k. Finally, we denote by η_I' the unique element of $\mathbb{C} \wedge^r_{\mathbb{Z}[\Gamma_I]} E_{L_I,S,T} \cdot e'_{r,S}$ satisfying

$$R_I'(\eta_I') = e'_{r,S} \cdot \theta^*_{L_I/k,S,T}(0).$$

The classical Rubin-Stark conjecture predicts that $\eta_I' \in \Lambda_{L_I/k,S,T}$.

Proposition 4.18. *The notation being as above, one has in $\mathbb{C} \wedge^r_{\mathbb{C}[G]} E_{K,S,T}$ the following equality:*

$$\eta_I = \frac{1}{|D_I|^r}\eta_I'.$$

Proof. First of all, by definition of η_I' we have

$$\eta_I' = e'_{r,S} \cdot \eta_I' = \mathrm{res}_{K/L_I}(e_I) \cdot \eta_I' = e_I \cdot \eta_I'.$$

Hence, we just have to show that

$$e_\chi \cdot \eta_I = e_\chi \cdot \frac{1}{|D_I|^r}\eta_I',$$

for all $\chi \in \widehat{G}_{r,S,I}$.

Fix such a character $\chi \in \widehat{G}_{r,S,I}$. On one hand, we have

$$R_I(e_\chi \cdot \eta_I) = e_\chi \cdot \theta^*_{K/k,S,T}(0).$$

On the other hand, using Lemma 4.9, (31), and (32), we have

$$\begin{aligned} R_I\left(e_\chi \cdot \frac{1}{|D_I|^r}\eta_I'\right) &= e_\chi \cdot \frac{1}{|D_I|^r} R_I(\eta_I') \\ &= e_\chi \cdot \frac{1}{|D_I|^r}|D_I|^{r-1}\mathrm{cor}_{K/L_I}(R_I'(\eta_I')) \\ &= e_\chi \cdot \frac{1}{|D_I|}\mathrm{cor}_{K/L_I}(e'_{r,S} \cdot \theta^*_{L_I/k,S,T}(0)) \\ &= e_\chi \cdot \frac{1}{|D_I|}\mathrm{cor}_{K/L_I}(\mathrm{res}_{K/L_I}(e_I \cdot \theta^*_{K/k,S,T}(0))) \\ &= e_\chi \cdot \frac{1}{|D_I|} N_I \cdot e_I \cdot \theta^*_{K/k,S,T}(0) \\ &= e_\chi \cdot \theta^*_{K/k,S,T}(0). \end{aligned}$$

Thus,

$$R_I(\eta_I) = R_I\left(\frac{1}{|D_I|^r}\eta_I'\right),$$

and since $R_I : \mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I \longrightarrow \mathbb{C}[G] \cdot e_I$ is an isomorphism of $\mathbb{C}[G]$-modules, we conclude the desired result. □

Let us suppose we are given a tower of number fields $k \subseteq M \subseteq K$, where K/k is abelian. Let $H = \mathrm{Gal}(K/M)$, $\Gamma = \mathrm{Gal}(M/k)$ and $G = \mathrm{Gal}(K/k)$. We start with the following useful lemma.

Lemma 4.19. *With the notation as above, we have an isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(E_{M,S,T}, \mathbb{Z}[G]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E_{M,S,T}, \mathbb{Z}[\Gamma]),$$

given by

$$\phi \mapsto \frac{1}{|H|} \cdot \mathrm{res}_{K/M} \circ \phi,$$

for $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{M,S,T}, \mathbb{Z}[G])$. The inverse map is given by

$$\phi \mapsto \mathrm{cor}_{K/M} \circ \phi,$$

whenever $\phi \in \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E_{M,S,T}, \mathbb{Z}[\Gamma])$.

Proof. First, we remark that the definition of the first morphism makes sense, since given $u \in E_{M,S,T}$ and $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{M,S,T}, \mathbb{Z}[G])$, one has

$$\mathrm{res}_{K/M} \circ \phi(u) \in |H| \cdot \mathbb{Z}[\Gamma].$$

The rest of the lemma follows from formulas (32) and (33). □

Since $\mathbb{Q}[G]$ is semi-simple, we have an injective morphism of $\mathbb{Q}[G]$-modules

$$\mathbb{Q} \bigwedge_{\mathbb{Z}[\Gamma]}^{r} E_{M,S,T} \hookrightarrow \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T},$$

which induces an injective $\mathbb{Z}[G]$-module morphism

$$\Lambda_{M/k,S,T} \hookrightarrow \Lambda_{K/k,S,T}.$$

In fact, more is true as the following lemma shows.

Lemma 4.20. *With the notation as above, the inclusion*

$$\mathbb{Q} \bigwedge_{\mathbb{Z}[\Gamma]}^{r} E_{M,S,T} \hookrightarrow \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}$$

induces the inclusion

$$\frac{1}{|H|^{r-1}} \cdot \Lambda_{M/k,S,T} \hookrightarrow \Lambda_{K/k,S,T}.$$

Proof. Let us start with $x \in \Lambda_{M/k,S,T}$ and $\phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S,T}, \mathbb{Z}[G])$, for $i = 1, \ldots, r$. Given $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S,T}, \mathbb{Z}[G])$, we denote its restriction to $E_{M,S,T}$ by the same symbol ϕ. Because of Lemma 4.19, we look at

$$\frac{1}{|H|} \mathrm{res}_{K/M} \circ \phi_i \in \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E_{M,S,T}, \mathbb{Z}[\Gamma]),$$

and by hypothesis, we know that

$$\left(\frac{1}{|H|} \mathrm{res}_{K/M} \circ \phi_1\right) \wedge \ldots \wedge \left(\frac{1}{|H|} \mathrm{res}_{K/M} \circ \phi_r\right)(x) \in \mathbb{Z}[\Gamma].$$

Applying the corestriction map and using formulas (31) and (32), we have

$$\begin{aligned}
&\mathrm{cor}_{K/M}\left(\left(\frac{1}{|H|} \mathrm{res}_{K/M} \circ \phi_1\right) \wedge \ldots \wedge \left(\frac{1}{|H|} \mathrm{res}_{K/M} \circ \phi_r\right)(x)\right) \\
&\quad = \frac{1}{|H|^{r-1}} \left(\frac{1}{|H|} \mathrm{cor}_{K/M} \circ \mathrm{res}_{K/M} \circ \phi_1\right) \wedge \ldots \wedge \left(\frac{1}{|H|} \mathrm{cor}_{K/M} \circ \mathrm{res}_{K/M} \circ \phi_r\right)(x) \\
&\quad = \frac{1}{|H|^{r-1}} \left(\frac{N_H}{|H|} \cdot \phi_1\right) \wedge \ldots \wedge \left(\frac{N_H}{|H|} \cdot \phi_r\right)(x) \\
&\quad = \frac{N_H}{|H|^r} \cdot \phi_1 \wedge \ldots \wedge \phi_r(x) \\
&\quad = \frac{1}{|H|^{r-1}} \cdot \phi_1 \wedge \ldots \wedge \phi_r(x) \in \mathbb{Z}[G].
\end{aligned}$$

It follows that

$$\frac{1}{|H|^{r-1}} \cdot \Lambda_{M/k,S,T} \hookrightarrow \Lambda_{K/k,S,T},$$

as we wanted to show. □

A discrepancy between the rank one and the higher order of vanishing cases is that the Rubin lattice does not behave well under taking fixed points, meaning that in general we have

$$\Lambda_{M/k,S,T} \subsetneq (\Lambda_{K/k,S,T})^H,$$

but nevertheless, the following lemma holds true:

Lemma 4.21. *The notation being as above, we have*

$$|H|^{r-1} \cdot (\Lambda_{K/k,S,T})^H \subseteq \Lambda_{M/k,S,T} \subseteq (\Lambda_{K/k,S,T})^H.$$

Proof. The right inclusion is clear.

We remark that

$$\left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}\right)^H = \mathbb{Q} \bigwedge_{\mathbb{Z}[\Gamma]}^{r} E_{M,S,T}.$$

Therefore,

$$\left(\Lambda_{K/k,S,T}\right)^H = \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[\Gamma]}^{r} E_{M,S,T}\right) \bigcap \Lambda_{K/k,S,T} \subseteq \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}.$$

Let us start with $x \in \left(\Lambda_{K/k,S,T}\right)^H$ and $\phi_i \in \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(E_{M,S,T}, \mathbb{Z}[\Gamma])$, for $i = 1, \ldots, r$. Because of Lemma 4.19, we will consider

$$\mathrm{cor}_{K/M} \circ \phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{M,S,T}, \mathbb{Z}[G]).$$

Since $E_{M,S,T}$ is $\mathbb{Z}$-free, the natural map

$$\mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S,T}, \mathbb{Z}[G]) \twoheadrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(E_{M,S,T}, \mathbb{Z}[G])$$

is surjective which allow us to choose lifts $\bar{\phi}_i$ of $\mathrm{cor}_{K/M} \circ \phi_i$, for $i = 1, \ldots, r$. Since $x \in \left(\Lambda_{K/k,S,T}\right)^H$, one has

$$\bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_r(x) \in \mathbb{Z}[G]^H.$$

Therefore, since $x \in \mathbb{Q} \wedge_{\mathbb{Z}[\Gamma]}^r E_{M,S,T}$, one has

$$\bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_r(x) = (\mathrm{cor}_{K/M} \circ \phi_1) \wedge \ldots \wedge (\mathrm{cor}_{K/M} \circ \phi_r)(x) \in \mathbb{Z}[G]^H.$$

Applying the restriction map and (33), we obtain

$$\begin{aligned} &\mathrm{res}_{K/M}\left((\mathrm{cor}_{K/M} \circ \phi_1) \wedge \ldots \wedge (\mathrm{cor}_{K/M} \circ \phi_r)(x)\right) \\ &\quad = (\mathrm{res}_{K/M} \circ \mathrm{cor}_{K/M} \circ \phi_1) \wedge \ldots \wedge (\mathrm{res}_{K/M} \circ \mathrm{cor}_{K/M} \circ \phi_r)(x) \\ &\quad = (|H| \cdot \phi_1) \wedge \ldots \wedge (|H| \cdot \phi_r)(x) \\ &\quad = |H|^r \cdot \phi_1 \wedge \ldots \wedge \phi_r(x) \in |H| \cdot \mathbb{Z}[\Gamma]. \end{aligned}$$

It follows that

$$|H|^{r-1} \cdot (\Lambda_{K/k,S,T})^H \subseteq \Lambda_{M/k,S,T},$$

as we wanted to show. ☐

Coming back to the setting of Proposition 4.18, we then have the following proposition.

Proposition 4.22. *With the notation as above, let us suppose that the Rubin-Stark conjecture is true for the data $(L_I/k, S, T, \{v_i \,|\, i \in I\})$, that is $\eta_I' \in \Lambda_{L_I/k,S,T}$. If Conjecture 4.16 is valid, then*

$$\eta_I \in \frac{1}{|D_I|^{r-1}} \cdot \Lambda_{L_I/k,S,T}.$$

Proof. By hypothesis, we have $\eta_I \in \Lambda_{K/k,S,T}$ and $\eta_I' \in \Lambda_{L_I/k,S,T}$. Proposition 4.18 implies that

$$|D_I|^r \cdot \eta_I = \eta_I'.$$

It follows that

$$\eta_I \in \left(\Lambda_{K/k,S,T}\right)^{D_I},$$

and by Lemma 4.21, we conclude that

$$|D_I|^{r-1} \cdot \eta_I \in \Lambda_{L_I/k,S,T}.$$

In other words,

$$\eta_I \in \frac{1}{|D_I|^{r-1}} \cdot \Lambda_{L_I/k,S,T}$$

as we wanted to show. □

This previous study of the behavior of the Rubin lattice in a tower of fields allow us to generalize Conjecture 4.12 to the higher order of vanishing situation.

Conjecture 4.23. *Let K/k be a finite abelian extension of number fields and S an r-cover. Suppose that $S \neq S_{min}$ and $|S| \geq r+2$. Let T be another finite set of finite primes of k satisfying Hypothesis 3.4. Let $\eta_I' \in \mathbb{C} \wedge^r_{\mathbb{Z}[\Gamma_I]} E_{L_I,S,T} \cdot e_{r,S}'$ be the unique element so that*

$$R_I'(\eta_I') = e_{r,S}' \cdot \theta^*_{L_I/k,S,T}(0),$$

where R_I' is the regulator at the level of L_I/k. Then

$$\eta_I' \in |D_I| \cdot \Lambda_{L_I/k,S,T}.$$

At last, we can now understand the precise relationship between Conjecture 4.16 and Conjecture 4.23.

Proposition 4.24. *Assuming the Rubin-Stark conjecture, Conjectures 4.16 and 4.23 are equivalent.*

Proof. Let us suppose first that Conjecture 4.16 is true. Then Proposition 4.18 implies that

$$\eta_I' = |D_I|^r \cdot \eta_I.$$

By Proposition 4.22, we conclude that

$$\eta_I' \in |D_I| \cdot \Lambda_{L_I/k,S,T},$$

hence Conjecture 4.23 holds true.

Conversely, if Conjecture 4.23 holds true, then $\eta_I' \in |D_I| \cdot \Lambda_{L_I/k,S,T}$ and Proposition 4.18 implies

$$\eta_I \in \frac{1}{|D_I|^{r-1}} \cdot \Lambda_{L_I/k,S,T}.$$

It follows from Lemma 4.20 that $\eta_I \in \Lambda_{K/k,S,T}$; hence, Conjecture 4.16 holds true. □

Remark. Let us suppose that S is an r-cover which contains r-places which split completely, say $v_1, \ldots, v_r$ and such that $|S| \geq r+2$. Then, either $S_{min} = \varnothing$ in which case S is in fact a $(r+1)$-cover, or $S_{min} = \{v_1, \ldots, v_r\}$ in which case Conjectures 4.15, 4.16, and 4.23 reduce to the classical Rubin-Stark conjecture. We note also, that if $r = 1$, then Conjecture 4.15, resp. Conjecture 4.16, reduces to Conjecture 4.5, resp. Conjecture 4.7, because of Lemma 4.14. Similarly, Conjecture 4.23 reduces to Conjecture 4.12 when $r = 1$.

5. Reduction to the case where $Cl_{K,S,T} = 1$

In §3.4, we made Hypothesis 3.5, that is we assumed that $Cl_{K,S,T} = 1$. If we want to show that the equivariant Tamagawa number conjecture implies Conjecture 4.16, then we need to know that it is enough to show Conjecture 4.16 when Hypothesis 3.5 is satisfied. This is the purpose of this section. The main idea is to throw some split primes into S in order to go to a higher order of vanishing situation. It goes back to Rubin in [23] and was also used by Burns in [6].

The following lemma is simple and the proof is left to the reader.

Lemma 5.1. *Let K/k be a finite abelian extension of number fields and S an r-cover. Suppose as usual that $|S| \geq r+2$ and $S \neq S_{min}$. Let T be another finite set of finite primes satisfying $S \cap T = \varnothing$ and $\mathfrak{p} \notin S \cup T$ be a finite prime of k which splits completely in K/k. Set $S' = S \cup \{\mathfrak{p}\}$. Then the following are true:*

(1) *S' is a $(r+1)$-cover,*
(2) *$S'_{min} = S_{min} \cup \{\mathfrak{p}\}$,*
(3) *$S' \neq S'_{min}$.*

This last lemma allows us to state the following proposition.

Proposition 5.2. *Let K/k be a finite abelian extension of number fields and S be an r-cover. Suppose also that $|S| \geq r+2$ and $S \neq S_{min}$. Let T be another finite set of finite primes satisfying Hypothesis 3.4. Let $\mathfrak{p} \notin S \cup T$ be a finite prime of k which splits completely in K/k and set $S' = S \cup \{\mathfrak{p}\}$. We fix $I \in \wp_r(S_{min})$ and we let*

$$I' = I \cup \{\mathfrak{p}\} \in \wp_{r+1}(S'_{min}).$$

If Conjecture 4.16 is true for the data $(K/k, S', T, I', r+1)$, then Conjecture 4.16 is true for the data $(K/k, S, T, I, r)$.

Proof. First of all, we remark that

$$\widehat{G}_{r,S,I} = \widehat{G}_{r+1,S',I'},$$

and thus $e_I = e_{I'}$. We fix a prime $\mathfrak{P}$ of K lying above $\mathfrak{p}$ and we let $f_{\mathfrak{P}} \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S',T}, \mathbb{Z}[G])$ be defined by

$$f_{\mathfrak{P}}(u) = \sum_{\sigma \in G} \mathrm{ord}_{\mathfrak{P}}(u^{\sigma}) \cdot \sigma^{-1}.$$

We note that

$$\ell_{\mathfrak{P}}(u) = f_{\mathfrak{P}}(u) \cdot \log \mathbb{N}(\mathfrak{p}).$$

A simple computation shows that the link between $R_{I'}$ and R_I is

$$R_{I'} = \varepsilon \cdot \log \mathbb{N}(\mathfrak{p}) R_I \circ \tilde{f}_{\mathfrak{P}}, \tag{36}$$

where $\varepsilon = \pm 1$ and

$$\tilde{f}_{\mathfrak{P}} : \bigwedge_{\mathbb{Z}[G]}^{r+1} E_{K,S',T} \longrightarrow \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S',T}$$

is the induced map explained in §3.4.1. The sign will be irrelevant for our purposes. Moreover, starting from the identity

$$\theta_{K/k,S',T}(s) = (1 - \sigma_{\mathfrak{p}}^{-1} \cdot \mathbb{N}(\mathfrak{p})^{-s}) \cdot \theta_{K/k,S,T}(s),$$

one has

$$\theta^{(r+1)}_{K/k,S',T}(0) = (r+1) \cdot \log \mathbb{N}(\mathfrak{p}) \cdot \theta^{(r)}_{K/k,S,T}(0),$$

since $\mathfrak{p}$ is assumed to split completely in K/k. It follows that

$$\begin{aligned} e_{I'} \cdot \theta^{*}_{K/k,S',T}(0) &= e_I \cdot \frac{\theta^{(r+1)}_{K/k,S',T}(0)}{(r+1)!} \\ &= e_I \cdot \frac{(r+1) \cdot \log \mathbb{N}(\mathfrak{p}) \cdot \theta^{(r)}_{K/k,S,T}(0)}{(r+1)!} \\ &= e_I \cdot \log \mathbb{N}(\mathfrak{p}) \frac{\theta^{(r)}_{K/k,S,T}(0)}{r!} \\ &= e_I \cdot \log \mathbb{N}(\mathfrak{p}) \cdot \theta^{*}_{K/k,S,T}(0). \end{aligned} \tag{37}$$

Putting (36) and (37) together, we get

$$e_I \cdot \theta^{*}_{K/k,S,T}(0) = R_I(\tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'})).$$

From the injective $\mathbb{Z}[G]$-module morphism $E_{K,S,T} \hookrightarrow E_{K,S',T}$, one obtains an injective $\mathbb{C}[G]$-module morphism

$$\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \hookrightarrow \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S',T},$$

since $\mathbb{C}[G]$ is semi-simple. If we show that

$$\tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'}) \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \cdot e_I,$$

we would then conclude that

$$\eta_I = \tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'}),$$

since $R_I : \mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I \longrightarrow \mathbb{C}[G] \cdot e_I$ is an isomorphism of $\mathbb{C}[G]$-modules. Let us remark that we just have to show

$$\tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'}) \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T},$$

since $e_{I'} \cdot \eta_{I'} = \eta_{I'}$ by definition of $\eta_{I'}$ and $e_I = e_{I'}$ as we noted before.

In fact,

$$\tilde{f}_{\mathfrak{P}} \left(\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r+1} E_{K,S',T} \right) \subseteq \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}, \tag{38}$$

and this can be seen as follows. Starting with the exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow E_{K,S,T} \longrightarrow E_{K,S',T} \longrightarrow \mathbb{Z}[G] \cdot \mathfrak{P} \longrightarrow C \longrightarrow 0,$$

where C is the subgroup of $Cl_{K,S,T}$ generated by the primes lying above $\mathfrak{p}$, we obtain the following short exact sequence of $\mathbb{C}[G]$-modules after tensoring with $\mathbb{C}$:

$$0 \longrightarrow \mathbb{C}E_{K,S,T} \longrightarrow \mathbb{C}E_{K,S',T} \longrightarrow \mathbb{C}[G] \longrightarrow 0.$$

We replaced $\mathbb{C}[G] \cdot \mathfrak{P}$ by $\mathbb{C}[G]$ in this last short exact sequence, since both $\mathbb{C}[G]$-modules are isomorphic, because $\mathfrak{p}$ is assumed to split completely in K/k. Let us choose a section $\iota : \mathbb{C}[G] \longrightarrow \mathbb{C}E_{K,S',T}$ of this last short exact sequence. Then, we have an isomorphism of $\mathbb{C}[G]$-modules

$$\left(\left(\bigwedge_{\mathbb{C}[G]}^{r} \mathbb{C}E_{K,S,T} \right) \otimes_{\mathbb{C}[G]} \iota(\mathbb{C}[G]) \right) \bigoplus \left(\bigwedge_{\mathbb{C}[G]}^{r+1} \mathbb{C}E_{K,S,T} \right) \xrightarrow[\simeq]{\alpha} \bigwedge_{\mathbb{C}[G]}^{r+1} \mathbb{C}E_{K,S',T},$$

defined as follows. On the first factor, it satisfies for $x \in \wedge^r_{\mathbb{C}[G]} \mathbb{C}E_{K,S,T}$ and $\lambda \in \mathbb{C}[G]$

$$x \otimes \iota(\lambda) \mapsto x \wedge \iota(\lambda).$$

On the second factor, it is just the map induced by the inclusion $\mathbb{C}E_{K,S,T} \hookrightarrow \mathbb{C}E_{K,S',T}$. Now, if $x \in \wedge^{r+1}_{\mathbb{C}[G]} \mathbb{C}E_{K,S,T}$, then

$$\tilde{f}_{\mathfrak{P}} \circ \alpha(x) = 0,$$

since $f_{\mathfrak{P}}$ is trivial on $E_{K,S,T}$. On the other hand, if $x \otimes \iota(\lambda)$ is in the first factor, then

$$\tilde{f}_{\mathfrak{P}} \circ \alpha(x \otimes \iota(\lambda)) = \pm f_{\mathfrak{P}}(\iota(\lambda)) \cdot x \in \bigwedge_{\mathbb{C}[G]}^{r} \mathbb{C}E_{K,S,T} \simeq \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T}.$$

It follows that claim (38) holds true and we have the equality $\eta_I = \tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'})$.

Let $\phi_1, \ldots, \phi_r \in E^*_{K,S,T}$. Since

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}),$$

as abelian groups for any $\mathbb{Z}[G]$-module M, and since $E_{K,S,T}$ is $\mathbb{Z}$-free, the map

$$\mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S',T}, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S,T}, \mathbb{Z}[G])$$

induced by the natural inclusion $E_{K,S,T} \hookrightarrow E_{K,S',T}$ is surjective. Let $\bar{\phi}_1, \ldots, \bar{\phi}_r$ be lifts of $\phi_1, \ldots, \phi_r$. We then have

$$\begin{aligned} \phi_1 \wedge \ldots \wedge \phi_r(\eta_I) &= \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_r(\eta_I) \\ &= \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_r(\tilde{f}_{\mathfrak{P}}(\varepsilon \cdot \eta_{I'})) \\ &= \varepsilon \cdot f_{\mathfrak{P}} \wedge \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_r(\eta_{I'}) \in \mathbb{Z}[G], \end{aligned}$$

since we assumed that Conjecture 4.16 for the data $(K/k, S', T, I', r+1)$ was true. We then conclude the desired result. □

Proposition 5.3. *It suffices to prove Conjecture 4.16 under the assumption* $Cl_{K,S,T} = 1$.

Proof. This is an application of Chebotarev's theorem. Indeed, if $Cl_{K,S,T} \neq 1$, then by Chebotarev's theorem, one can find finitely many split primes not in $S \cup T$, say $v_{n+1}, \dots, v_{n+t}$, so that $Cl_{K,S',T} = 1$ where

$$S' = S \cup \{v_{n+1}, \dots, v_{n+t}\}.$$

One obtains the desired result by successive application of Proposition 5.2. □

6. The equivariant Tamagawa number conjecture and the extended abelian Stark conjecture

In this section, we show that the equivariant Tamagawa number conjecture implies Conjecture 4.16. In fact, we adapt Burns's proof that the Rubin-Stark conjecture follows from the equivariant Tamagawa number conjecture contained in [6].

6.1. **Preliminaries.** We start by introducing some notation. If R is a commutative ring with 1, then $M_d(R)$ denotes the ring of d by d matrices with coefficients in R. The group consisting of invertible matrices will be denoted by $GL_d(R)$. In this section, we shall use the letters I, J, K, L, M to denote elements of $\wp_r(d)$ for some integers r and d with $1 \leq r \leq d$. The letters K, L, M are usually used to denote number fields, but we think that this ambiguity should not cause any confusion.

Given $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r) \in \wp_r(d)$, we define

$$(-1)^{I+J} := (-1)^{i_1 + \dots + i_r + j_1 + \dots + j_r}.$$

If $I \in \wp_r(d)$, then we let $I^* = (1, 2, \dots, d) \smallsetminus I \in \wp_{d-r}(d)$.

Following Burns, it will be convenient to work with certain permutations. If $I, J \in \wp_r(d)$ and $J \subseteq I$, then we let $\tau_{I,J}$ denote the permutation $I \mapsto J \cdot (I \smallsetminus J)$, where $\cdot$ means concatenation. Given an integer $t \in \mathbb{Z}_{\geq 1}$, we let

$$[t] = (1, 2, \dots, t),$$

which we view as an element of $\wp_t(d)$ if $t \leq d$. Suppose that s and t are integers satisfying $1 \leq s \leq t \leq d$ and define the permutation $\tau_{s,t}$ to be the cycle

$$\tau_{s,t} = (t \;\; t-1 \;\; t-2 \;\; \dots \;\; s+1 \;\; s).$$

One has

$$\mathrm{sgn}(\tau_{s,t}) = (-1)^{s+t}.$$

Now, if $I = (i_1, \dots, i_r) \in \wp_r(d)$, then

$$\tau_{[d],I} = \tau_{1,i_1} \circ \tau_{2,i_2} \circ \dots \circ \tau_{r,i_r}.$$

Thus

$$\mathrm{sgn}(\tau_{[d],I}) = (-1)^{I+[r]},$$

and moreover given another $J \in \wp_r(d)$

$$(-1)^{I+J} = (-1)^{I+J+2\cdot[r]} = \mathrm{sgn}(\tau_{[d],I}) \cdot \mathrm{sgn}(\tau_{[d],J}).$$

Let $A \in M_d(R)$. If $I, J \in \wp_r(d)$, then we denote by $A^{(J,I)}$ the matrix obtained from A by deleting the j-th row for $j \in J$ and the i-th column for $i \in I$. With these notations, the Laplace expansion of A along the column set $I \in \wp_r(d)$ is given by

$$\begin{aligned}\det(A) &= \sum_{J \in \wp_r(d)} (-1)^{J+I} \det(A^{(J,I)}) \cdot \det(A^{(J^*,I^*)}) \\ &= \mathrm{sgn}(\tau_{[d],I}) \sum_{J \in \wp_r(d)} \mathrm{sgn}(\tau_{[d],J}) \cdot \det(A^{(J,I)}) \cdot \det(A^{(J^*,I^*)}).\end{aligned}$$

If $r = 1$, this last formula is just the usual expansion of the determinant along a single column, and the general formula follows from the case $r = 1$ by induction.

Given two matrices $A, B \in M_d(R)$ and $I \in \wp_r(d)$, we let $A(I, B)$ denote the matrix obtained form A when replacing the i-th column of A by the i-th column of B for $i \in I$.

If $I = (i_1, \ldots, i_r) \in \wp_r(d)$ is given, then we also use the notation

$$I_u = i_u,$$

for $u = 1, \ldots, r$. This notation will be used if we do not want to enumerate the elements of I.

6.2. **The equivariant Tamagawa number conjecture implies Conjecture 4.16.** As usual, K/k is a finite abelian extension of number fields with Galois group G and we place ourselves in the setting of Conjecture 4.16. That is, we are given a finite set of primes

$$S = \{v_0, v_1, \ldots, v_n\},$$

containing $S(K/k)$ which is an r-cover for K/k such that $S \neq S_{min}$ and $|S| \geq r + 2$. Let us remark that if $S_{min} \neq \varnothing$, then $|S_{min}| \geq r$ necessarily. If $S_{min} = \varnothing$, there is nothing to prove. We suppose that

$$S_{min} = \{v_1, \ldots, v_m\},$$

for some m satisfying $r \leq m \leq n$. We fix $I \in \wp_r(m)$ and we suppose given a finite set of finite primes T of k satisfying Hypothesis 3.4. Proposition 5.3 allows us to make the following hypothesis which we shall make throughout this subsection (§6.2).

Hypothesis 6.1. *The (S, T)-class group is trivial: $Cl_{K,S,T} = 1$.*

The main point is that we need this hypothesis in order to apply the results of §3.4. Let us start with a simple, but useful lemma.

Lemma 6.2. *Let K/k be a finite abelian extension of number fields and S a finite set of primes of k. We write as usual $S = \{v_0, v_1, \ldots, v_n\}$. If $\chi \in \widehat{G}$, $\chi \neq \chi_1$, and j is an integer with $0 \leq j \leq n$ such that $G_j \not\subseteq \mathrm{Ker}(\chi)$, then*

$$e_\chi \cdot w_j = 0,$$

in $\mathbb{C}Y_{K,S}$.

Proof. Letting $\{\sigma_1, \ldots, \sigma_s\}$ be a complete set of representatives for G/G_j, we have

$$\begin{aligned} e_\chi \cdot w_j &= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \cdot w_j \\ &= \frac{1}{|G|} \sum_{t=1}^{s} \sum_{h \in G_j} \chi(\sigma_t h)(\sigma_t h)^{-1} \cdot w_j \\ &= \frac{1}{|G|} \sum_{t=1}^{s} \chi(\sigma_t) \left(\sum_{h \in G_j} \chi(h) \right) \sigma_t^{-1} \cdot w_j \\ &= 0, \end{aligned}$$

since $G_j \not\subseteq \mathrm{Ker}(\chi)$. □

As a corollary of the last lemma, we obtain:

Corollary 6.3. *If $\chi \in \widehat{G}_{r,S,I}$, then*

$$\mathbb{C}X_{K,S} \cdot e_\chi = \mathbb{C}Y_{K,S} \cdot e_\chi = \bigoplus_{i \in I} \mathbb{C}w_i \cdot e_\chi.$$

Hence, the elements $w_i \cdot e_\chi$ for $i \in I$ form a $\mathbb{C}$-basis of $\mathbb{C}X_{K,S} \cdot e_\chi$.

Proof. It suffices to remark that if $\chi \in \widehat{G}_{r,S,I}$, then $\dim_{\mathbb{C}}(\mathbb{C}X_{K,S} \cdot e_\chi) = r$ by Theorem 4.1. (Moreover, $\chi \neq \chi_1$, since we assumed that $|S| \geq r + 2$.) □

The η_I of Conjecture 4.16 can also be defined as follows. If $I = (i_1, \ldots, i_r) \in \wp_r(m)$ where

$$i_1 < \ldots < i_r,$$

then we let

$$w_I = w_{i_1} \wedge \ldots \wedge w_{i_r} \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} Y_{K,S}.$$

Let us remark that

$$\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} Y_{K,S} \cdot e_I = \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} X_{K,S} \cdot e_I,$$

since $\chi_1 \notin \widehat{G}_{r,S,I}$. The Dirichlet logarithm $\lambda : \mathbb{C}E_{K,S,T} \xrightarrow{\simeq} \mathbb{C}X_{K,S}$ induces an isomorphism of $\mathbb{C}[G]$-modules

$$\wedge^r \lambda : \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \xrightarrow{\simeq} \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} X_{K,S} \simeq \bigwedge_{\mathbb{C}[G]}^{r} \mathbb{C}X_{K,S},$$

defined on pure wedges by the formula

$$u_1 \wedge \ldots \wedge u_r \mapsto \lambda(u_1) \wedge \ldots \wedge \lambda(u_r).$$

Let

$$\eta'_I \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_{K,S,T} \cdot e_I$$

be the unique element so that

$$\wedge^r \lambda(\eta'_I) = e_I \cdot \theta^*_{K/k,S,T}(0) \cdot w_I.$$

We now claim that $\eta_I = \eta'_I$. Indeed, by Lemma 6.2, we have

$$\begin{aligned}
e_I \cdot \wedge^r \lambda(u_1 \wedge \ldots \wedge u_r) &= e_I \cdot \lambda(u_1) \wedge \ldots \wedge \lambda(u_r) \\
&= e_I \cdot \left(\sum_{i \in I} \ell_i(u_1) \cdot w_i \right) \wedge \ldots \wedge \left(\sum_{i \in I} \ell_i(u_r) \cdot w_i \right) \\
&= e_I \cdot \det(\ell_i(u_j)) \cdot w_I \\
&= e_I \cdot R_I(u_1 \wedge \ldots \wedge u_r) \cdot w_I.
\end{aligned}$$

Therefore,

$$\wedge^r \lambda(\eta_I) = R_I(\eta_I) \cdot w_I.$$

Hence, by definition of η_I, we have

$$\begin{aligned}
\wedge^r \lambda(\eta'_I) &= e_I \cdot \theta^*_{K/k,S,T}(0) \cdot w_I \\
&= R_I(\eta_I) \cdot w_I \\
&= \wedge^r \lambda(\eta_I).
\end{aligned}$$

Since $\wedge^r \lambda$ is an isomorphism of $\mathbb{C}[G]$-modules, we conclude that $\eta_I = \eta'_I$. Summarizing, η_I can be viewed as the unique element of $\mathbb{C} \wedge^r_{\mathbb{Z}[G]} E_{K,S,T} \cdot e_I$ satisfying either

$$R_I(\eta_I) = e_I \cdot \theta^*_{K/k,S,T}(0),$$

or

$$\wedge^r \lambda(\eta_I) = e_I \cdot \theta^*_{K/k,S,T}(0) \cdot w_I.$$

We will now apply the results of §3.4, more precisely Corollary 3.11 with a particular choice of ψ, ι_1, and ι_2. Let us fix an integer z. As we noted in §3.4 the two modules $\mathbb{Q}\Omega_{S,T}$ and $\mathbb{Q}F$ are isomorphic as $\mathbb{Q}[G]$-modules. Hence, given any prime p and tensoring with $\mathbb{Q}_p$, there is an isomorphism of $\mathbb{Q}_p[G]$-modules $\mathbb{Q}_p\Omega_{S,T} \simeq \mathbb{Q}_pF$. Both $\mathbb{Z}_p[G]$-modules $\mathbb{Z}_p\Omega_{S,T}$ and $\mathbb{Z}_pF$ being $\mathbb{Z}_p[G]$-projective, it follows from Swan's theorem that

$$\mathbb{Z}_p\Omega_{S,T} \simeq \mathbb{Z}_pF,$$

as $\mathbb{Z}_p[G]$-modules. Applying Roiter's lemma, we can choose an injective $\mathbb{Z}[G]$-module morphism

$$\psi' : F \hookrightarrow \Omega_{S,T},$$

such that the index $(\Omega_{S,T} : \psi'(F))$ is finite and

$$\gcd((\Omega_{S,T} : \psi'(F)), z \cdot |G|) = 1.$$

As in §3.4, we obtain an endomorphism $\phi' \in \mathrm{End}_{\mathbb{Z}[G]}(F)$. (See 18.) If the readers need to review Swan's theorem and Roiter's lemma, they can consult [10].

The next lemma is of technical nature and is necessary for the proof of Lemma 6.9 below. We present here a slight modification of Lemma 7.5 of [6]. The rank one situation is simpler and we do not need Lemma 6.4, since Lemma 6.9 below does not say anything useful when $r = 1$.

Lemma 6.4. *Let G be a finite abelian group and F a finitely generated free $\mathbb{Z}[G]$-module of $\mathbb{Z}[G]$-rank d. Let $\{b_1, \ldots, b_d\}$ be a $\mathbb{Z}[G]$-basis for F and let $\mu \in \mathrm{End}_{\mathbb{Z}[G]}(F)$. We let r be an integer satisfying $1 \le r < d$ and we suppose that $\Lambda \subseteq \widehat{G}$ is such that $\chi_1 \notin \Lambda$ and for all $\chi \in \Lambda$, one has*

$$\dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\mu) \cdot e_\chi) = r.$$

We also suppose that

$$\mathbb{C}\mathrm{Im}(\mu) \cdot e_\chi = \left(\bigoplus_{a=r+1}^{d} \mathbb{C}[G] \cdot b_a \right) \cdot e_\chi,$$

for all $\chi \in \Lambda$. We identify $\mathrm{Aut}_{\mathbb{Z}[G]}(F)$ with $GL_d(\mathbb{Z}[G])$ via the $\mathbb{Z}[G]$-basis $\{b_1, \ldots, b_d\}$. Then there exists $\varepsilon \in GL_d(\mathbb{Z})$ such that $\det(\varepsilon) = 1$ and

$$(\mathbb{C}\mathrm{Ker}(\mu \circ \varepsilon) \cdot e_\chi) \cap (\mathbb{C}\mathrm{Im}(\mu \circ \varepsilon) \cdot e_\chi) = 0,$$

for all $\chi \in \Lambda$.

Proof. We follow Burns closely. For each integer i with $r < i \le d$ and for each $\chi \in \Lambda$, we let

$$N_{i,\chi} = \bigoplus_{\substack{a=r+1 \\ a \ne i}}^{d} \mathbb{C} b_a \cdot e_\chi.$$

We shall prove the existence of ε inductively. Indeed, it suffices to show, for $i = 0, \ldots, d-r$, the existence of a $\varepsilon_i \in GL_d(\mathbb{Z})$ satisfying the following properties:

(1) $\det(\varepsilon_i) = 1$.
(2) For $i = 0, \ldots, d$, we let $\mu_i = \mu \circ \varepsilon_0 \circ \ldots \circ \varepsilon_i$. Then

$$\mathbb{C}\mathrm{Im}(\mu_i) \cdot e_\chi = \mathbb{C}\mathrm{Im}(\mu) \cdot e_\chi = \sum_{a=r+1}^{d} \mathbb{C}[G] \cdot b_a \cdot e_\chi,$$

for all $\chi \in \Lambda$,
(3) Given any $\chi \in \Lambda$ and

$$\sum_{a=r+1}^{d} z_{a,\chi} b_a \cdot e_\chi \in \mathrm{Ker}(\mu_i^\chi),$$

where $z_{a,\chi} \in \mathbb{C}$, then $z_{a,\chi} = 0$ for all a with $r < a \le r + i$.

For $i = 0$, we just take ε_0 to be the identity matrix. Then the three properties above for $i = 0$ are clearly satisfied. We now fix an integer j with $0 < j \le d - r$ and assume that for each integer i with $0 \le i < j$, an automorphism ε_i satisfying the three properties above has been constructed.

We then have to find $\varepsilon_j \in GL_d(\mathbb{Z})$ satisfying properties (1) and (3) above, the condition (2) being satisfied, since composing with an automorphism does not change the image.

For each integer i with $1 \le i \le r$ we define $\Lambda_{i,j}$ to be the set of $\chi \in \Lambda$ for which $\mathrm{Ker}(\mu_{j-1}^\chi)$ does not contain any element of the form $e_\chi \cdot b_i + \alpha$ for some $\alpha \in N_{r+j,\chi}$.

Following Burns, we now use these characters to define rational integers $\{c_{s,j} \mid r \ge s \ge 1\}$ and then we use these integers in order to define ε_j.

The integers $c_{s,j}$ are defined recursively as follows. First, if $s = r$, then for each $\chi \in \Lambda_{r,j}$, we let $Z_{r,j,\chi}$ denote the set of integers m such that $\mathrm{Ker}(\mu_{j-1}^{\chi})$ contains an element of the form

$$x_m := mb_r \cdot e_\chi + b_{r+j} \cdot e_\chi + \alpha_m,$$

for some $\alpha_m \in N_{r+j,\chi}$. Note that the set $Z_{r,j,\chi}$ can be empty. Moreover, $|Z_{r,j,\chi}| \leq 1$. Indeed, if $m, m' \in Z_{r,j,\chi}$, then

$$x_m - x_{m'} = (m - m')b_r \cdot e_\chi + (\alpha_m - \alpha_{m'}) \in \mathrm{Ker}(\mu_{j-1}^{\chi}).$$

By definition of $\Lambda_{r,j}$, we conclude that $m = m'$ necessarily. We can thus fix an integer $c_{r,j}$ which is greater than every element of the finite set

$$\bigcup_{\chi \in \Lambda_{r,j}} Z_{r,j,\chi}.$$

We now suppose that $c_{s,j}$ has been fixed for each s with $t < s \leq r$ and then choose $c_{t,j}$ as follows. Similarly as before, for each $\chi \in \Lambda_{t,j}$, we let $Z_{t,j,\chi}$ be the set of integers m such that $\mathrm{Ker}(\mu_{j-1}^{\chi})$ contains an element of the form

$$x_m := mb_t e_\chi + \sum_{i=t+1}^{r} c_{i,j} b_i \cdot e_\chi + b_{r+j} \cdot e_\chi + \alpha_m,$$

for some $\alpha_m \in N_{r+j,\chi}$. As above, $Z_{t,j,\chi}$ is either empty or of cardinality precisely one. We can thus choose an integer $c_{t,j}$ which is greater than every element of the finite set

$$\bigcup_{\chi \in \Lambda_{t,j}} Z_{t,j,\chi}.$$

We now define ε_j to be the automorphism of F which sends

$$b_a \mapsto \begin{cases} b_a + \sum_{i=1}^{r} c_{i,j} b_i, & \text{if } a = r + j, \\ b_a, & \text{if } 1 \leq a \leq d \text{ and } a \neq r + j. \end{cases}$$

The matrix of this automorphism with respect to the basis $\{b_1, \ldots, b_d\}$ is lower triangular with ones on the diagonal; therefore, $\det(\varepsilon_j) = 1$. Moreover, by its very definition it has coefficients in $\mathbb{Z}$.

We are now left to show that ε_j satisfies property (3) above. Let $\chi \in \Lambda$ and let

$$x = \sum_{a=r+1}^{d} z_{a,\chi} b_a \cdot e_\chi \in \mathrm{Ker}(\mu_j^{\chi}).$$

We have to show that $z_{a,\chi} = 0$ for $a = r + 1, \ldots, r + j$, but note that it suffices to show $z_{r+j,\chi} = 0$. Indeed, this follows from the induction hypothesis, namely property (3) above for $j - 1$.

We now suppose for the sake of contradiction that $z_{r+j,\chi} \neq 0$ and consider

$$y = z_{r+j,\chi}^{-1} \varepsilon_j(x) \in \mathrm{Ker}(\mu_{j-1}^{\chi}).$$

By definition of ε_j, we have

$$\begin{aligned} y &= z_{r+j,\chi}^{-1} \varepsilon_j(x) \\ &= \sum_{i=1}^{r} c_{i,j} b_i \cdot e_\chi + b_{r+j} \cdot e_\chi + \alpha', \end{aligned}$$

where

$$\alpha' = \sum_{\substack{a=r+1 \\ a \neq r+j}}^{d} z_{r+j,\chi}^{-1} z_{a,\chi} b_a \cdot e_\chi \in N_{r+j,\chi}.$$

If $\chi \notin \cup_{i=1}^{r} \Lambda_{i,j}$, then for each i with $1 \leq i \leq r$, there exists $\alpha_i \in N_{r+j,\chi}$ such that

$$b_i \cdot e_\chi + \alpha_i \in \mathrm{Ker}(\mu_{j-1}^{\chi}).$$

But $y \in \mathrm{Ker}(\mu_{j-1}^{\chi})$ and is not in the $\mathbb{C}$-linear span of $\{b_i \cdot e_\chi + \alpha_i \,|\, 1 \leq i \leq r\}$. Therefore,

$$\dim_{\mathbb{C}}(\mathrm{Ker}(\mu_{j-1}^{\chi})) > r.$$

Since $\dim_{\mathbb{C}}(\mathrm{Ker}(\mu_{j-1}^{\chi})) = \dim_{\mathbb{C}}(\mathrm{Ker}(\mu^{\chi}))$, this is a contradiction with the fact that $\chi \in \Lambda$.

Hence, we necessarily have $\chi \in \cup_{i=1}^{r} \Lambda_{i,j}$. Let i_0 be the smallest integer i satisfying $1 \leq i \leq r$ and $\chi \in \Lambda_{i,j}$. If $i_0 = 1$, then the containment $y \in \mathrm{Ker}(\mu_{j-1}^{\chi})$ is a contradiction with the fact that $c_{1,j} \notin Z_{1,j,\chi}$. If $i_0 > 1$, then for each integer i with $1 \leq i < i_0$, there is an element in $\mathrm{Ker}(\mu_{j-1}^{\chi})$ of the form $y_i = b_i \cdot e_\chi + \alpha_i$ for some $\alpha_i \in N_{r+j,\chi}$, since $\chi \notin \Lambda_{i,j}$. It follows that $\mathrm{Ker}(\mu_{j-1}^{\chi})$ contains the element

$$\begin{aligned} y' &= y - \sum_{i=1}^{i_0-1} c_{i,j} y_i \\ &= \sum_{i=i_0}^{r} c_{i,j} b_i e_\chi + b_{r+j} \cdot e_\chi + \left(\alpha' - \sum_{i=1}^{i_0-1} c_{i,j}\alpha_i\right). \end{aligned}$$

Since

$$\alpha' - \sum_{i=1}^{i_0-1} c_{i,j}\alpha_i \in N_{r+j,\chi},$$

this is a contradiction with the fact that $c_{i_0,j} \notin Z_{i_0,j,\chi}$. We are now done with the proof. □

We now apply this lemma with $\mu = \phi'$ and $\Lambda = \widehat{G}_{r,S,I}$. Note that since we suppose $|S| \geq r+2$, we necessarily have $d \geq n \geq r+1 > r$. In order to be allowed to apply the lemma, we observe:

Lemma 6.5. *For all $\chi \in \widehat{G}_{r,S,I}$, one has*

$$\mathbb{C}\mathrm{Im}(\phi') \cdot e_\chi = \left(\bigoplus_{i \notin I} \mathbb{C}[G] \cdot b_i\right) e_\chi.$$

Proof. This can be seen as follows. Remark that

$$\mathbb{C}\mathrm{Im}(\phi') \cdot e_\chi = \mathbb{C}\mathrm{Ker}(\pi) \cdot e_\chi.$$

Let

$$x = \sum_{i=1}^{d} z_i b_i \cdot e_\chi \in \mathbb{C}\mathrm{Ker}(\pi) \cdot e_\chi,$$

for some $z_i \in \mathbb{C}$. Then

$$\begin{aligned} 0 &= \pi(x) \\ &= \sum_{i \in I} z_i (w_i - w_0) \cdot e_\chi \\ &= \sum_{i \in I} z_i \cdot w_i \cdot e_\chi. \end{aligned}$$

Since, $\{w_i \cdot e_\chi \,|\, i \in I\}$ is a $\mathbb{C}$-basis for $\mathbb{C}X_{K,S} \cdot e_\chi = \mathbb{C}Y_{K,S} \cdot e_\chi$ by Corollary 6.3, we deduce that $z_i = 0$ for all $i \in I$. It follows that

$$x \in \left(\bigoplus_{i \notin I} \mathbb{C}[G] \cdot b_i\right) \cdot e_\chi,$$

and

$$\mathbb{C}\mathrm{Im}(\phi') \cdot e_\chi \subseteq \left(\bigoplus_{i \notin I} \mathbb{C}[G] \cdot b_i\right) e_\chi,$$

from which we conclude the equality by a dimension count. □

Thus, all the hypotheses of Lemma 6.4 are satisfied and we obtain an $\varepsilon \in GL_d(\mathbb{Z})$ having the properties described in the lemma.

Letting $\phi = \phi' \circ \varepsilon$ and $\psi = \psi' \circ \varepsilon$, we still have

$$((\Omega_{S,T} : \psi(F)), z \cdot |G|) = 1.$$

This ϕ now satisfies

$$(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi) \cap (\mathbb{C}\mathrm{Im}(\phi) \cdot e_\chi) = 0,$$

for all $\chi \in \widehat{G}_{r,S,I}$. Furthermore, we still have

$$\mathbb{C}\mathrm{Im}(\phi) \cdot e_\chi = \left(\bigoplus_{i \notin I} \mathbb{C}[G] \cdot b_i \right) e_\chi,$$

since the image of ϕ is the same as of ϕ'.

We now choose a section ι_2 satisfying

$$\iota_2(e_I \cdot w_i) = e_I \cdot b_i,$$

for all $i \in I$. This is legitimate, since $\pi(e_I \cdot b_i) = e_I \cdot (w_i - w_0) = e_I \cdot w_i$. We then apply the procedure explained in §3.4 and we obtain an endomorphism $f \in \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}F)$ and an element $x_T \in \mathbb{C}[G]^\times$ satisfying

$$\theta^*_{K/k,S,T}(0) = x_T \cdot \det(f). \tag{39}$$

Moreover, by Corollary 3.11, we know that $x_T \in \mathbb{Q}[G]$ and $x_T \in \mathbb{Z}_\ell[G]$ for all $\ell \mid z$.

Since we are interested in the special value $e_I \cdot \theta^*_{K/k,S,T}(0)$, we will now define an endomorphism $\kappa_I \in \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}F)$ which satisfies

$$e_I \cdot \det(f) = \det(\kappa_I).$$

We define κ_I as follows. We let

$$\rho_I : \mathbb{C}Y_{K,S} \twoheadrightarrow \bigoplus_{i \in I} \mathbb{C}[G] \cdot w_i,$$

be the natural projection and we define

$$\tilde{\rho}_I : \mathbb{C}Y_{K,S} \longrightarrow \bigoplus_{i \in I} \mathbb{C}[G] \cdot w_i,$$

as follows:

(1) $\tilde{\rho}_I = \rho_I$ on $\mathbb{C}Y_{K,S} \cdot e_I$,
(2) $\tilde{\rho}_I = 0$ on $\mathbb{C}Y_{K,S} \cdot (1 - e_I)$.

Moreover, we let

$$\tilde{\iota}_I : \mathbb{C}Y_{K,S} \longrightarrow \bigoplus_{i \in I} \mathbb{C}[G] \cdot b_i \subseteq \mathbb{C}F,$$

be the $\mathbb{C}[G]$-module morphism such that

(1) $\tilde{\iota}_I = \iota_2$ on $\mathbb{C}Y_{K,S} \cdot e_I = \mathbb{C}X_{K,S} \cdot e_I$,
(2) $\tilde{\iota}_I = 0$ on $\mathbb{C}Y_{K,S} \cdot (1 - e_I)$.

We then define

$$\kappa_I := \tilde{\iota}_I \circ \tilde{\rho}_I \circ \lambda \circ \psi \circ p + \phi \in \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}F),$$

where

$$p : \mathbb{C}F \twoheadrightarrow \mathbb{C}\mathrm{Ker}(\phi)$$

is the natural projection induced by the decomposition (26).

Lemma 6.6. *With the same notation as above, one has*

$$e_I \cdot \det(f) = \det(\kappa_I).$$

Proof. We show that if $\chi \in \widehat{G}_{r,S,I}$ then $\kappa_I^\chi = f^\chi$, whereas if $\chi \notin \widehat{G}_{r,S,I}$, then κ_I^χ is singular. Let us start with $\chi \in \widehat{G}_{r,S,I}$. On $\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi$, the map p^χ is just the identity map and so is $\tilde{\rho}_I^\chi$ on $\mathbb{C}Y_{K,S} \cdot e_\chi$. Thus on $\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi$, we have

$$\kappa_I^\chi = \tilde{\iota}_I^\chi \circ \lambda^\chi \circ \psi^\chi = \iota_2^\chi \circ \lambda^\chi \circ \psi^\chi = f^\chi.$$

On the other hand, on $\iota_1(\mathbb{C}\mathrm{Im}(\phi) \cdot e_\chi)$, the map p^χ is trivial; therefore,

$$\kappa_I^\chi = \phi^\chi = f^\chi.$$

Let now $\chi \notin \widehat{G}_{r,S,I}$, then the map $\tilde{\rho}_I^\chi$ is the trivial map by definition, and we have

$$\kappa_I^\chi = \phi^\chi,$$

but ϕ^χ is singular, since S is an r-cover and $r \geq 1$. Indeed,

$$\dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi) \geq r.$$

This finishes the proof. □

Starting with (39), we have

$$\begin{aligned} e_I \cdot \theta^*_{K/k,S,T}(0) &= e_I \cdot x_T \cdot \det(f) \\ &= x_T \cdot \det(\kappa_I). \end{aligned}$$

We now write the matrix of κ_I with respect to the basis $\{b_1, \ldots, b_d\}$. We denote the matrix corresponding to ϕ by $B = (b_{st})$. By definition, one has

$$\phi(b_s) = \sum_{t=1}^{d} b_{st} b_t,$$

for $s = 1, \ldots, d$. We have $B \in M_d(\mathbb{Z}[G])$.

Lemma 6.7. *With the same notation as above,* $\det(B) = 0$ *and if* $K, L \in \wp_s(d)$*, where* $s < r$*, then*

$$\det\left(B^{(K,L)}\right) = 0.$$

Proof. We let $\phi_{K,L}$ be the composition of the two following maps

$$\bigoplus_{k \notin K} \mathbb{C}[G] \cdot b_k \xrightarrow{\phi_{res}} \mathbb{C}F \longrightarrow \bigoplus_{l \notin L} \mathbb{C}[G] \cdot b_l,$$

where ϕ_{res} denotes the restriction of the map ϕ to the space

$$\bigoplus_{k \notin K} \mathbb{C}[G] \cdot b_k.$$

The morphism $\phi_{K,L}$ corresponds to the matrix $B^{(K,L)}$. In order to conclude the desired result, we just have to show that $\phi_{K,L}^\chi$ is singular for all $\chi \in \widehat{G}$. We shall show that

$$(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi) \bigcap \left(\bigoplus_{k \notin K} \mathbb{C}[G] \cdot b_k\right) \cdot e_\chi \neq 0.$$

If not, we would have

$$\begin{aligned} \dim_{\mathbb{C}}\left(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi + \left(\bigoplus_{k \notin K} \mathbb{C}[G] \cdot b_k\right) \cdot e_\chi\right) &= \dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi) + d - s \\ &\geq r + d - s \\ &> d, \end{aligned}$$

since S is an r-cover and this would be a contradiction. □

For $i \in I$, let $\zeta_i \in \mathbb{C}[G]$ be such that

$$\tilde{\iota}_I(w_i) = \zeta_i \cdot b_i.$$

Remark that $e_I \cdot \zeta_i = e_I$, because of the choice of ι_2. For $s = 1, \dots, d$, we choose $a'_{si} \in \mathbb{C}[G]$ so that

$$\tilde{\rho}_I \circ \lambda \circ \psi \circ p(b_s) = \sum_{i \in I} a'_{si} \cdot w_i.$$

Applying $\tilde{\iota}_I$ to this last equation, we get

$$\tilde{\iota}_I \circ \tilde{\rho}_I \circ \lambda \circ \psi \circ p(b_s) = \sum_{i \in I} \zeta_i \cdot a'_{si} \cdot b_i.$$

We set

$$a_{si} = \zeta_i \cdot a'_{si},$$

and we define the matrix $A_I = (a_{st}) \in M_d(\mathbb{C}[G])$, where

$$a_{st} = \begin{cases} a_{si}, & \text{if } t = i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

For future use, we also let $A'_I = (a'_{st}) \in M_d(\mathbb{C}[G])$, where

$$a'_{st} = \begin{cases} a'_{si}, & \text{if } t = i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix corresponding to κ_I is then

$$C_I = A_I + B.$$

We will now take the Laplace expansion of the determinant of C_I along the column set I.

$$\begin{aligned} e_I \cdot \theta^*_{K/k,S,T}(0) \cdot w_I &= x_T \cdot \det(C_I) \cdot w_I \\ &= x_T \sum_{J \in \wp_r(d)} (-1)^{I+J} \det\left(C_I^{(J,I)}\right) \cdot \det\left(C_I^{(J^*,I^*)}\right) \cdot w_I \\ &= x_T \sum_{J \in \wp_r(d)} (-1)^{I+J} \det\left(B^{(J,I)}\right) \cdot \det\left((A_I + B)^{(J^*,I^*)}\right) \cdot w_I \\ &= x_T \sum_{J \in \wp_r(d)} (-1)^{I+J} \det\left(B^{(J,I)}\right) \cdot \det\left(A_I^{(J^*,I^*)} + B^{(J^*,I^*)}\right) \cdot w_I. \end{aligned}$$

We have

$$\det\left(A_I^{(J^*,I^*)} + B^{(J^*,I^*)}\right) = \sum_{K \in \wp(I)} \det\left(A_I^{(J^*,I^*)}\left(K, B^{(J^*,I^*)}\right)\right),$$

where

$$\wp(I) = \bigcup_{t=0}^{r} \wp_t(I).$$

(The set $\wp_t(I)$, where t is an integer and $I \in \wp_r(d)$, is defined similarly as $\wp_t(d)$, where $d \in \mathbb{Z}_{\geq 1}$.) We remark that if $K \neq \varnothing$, then

$$\sum_{J \in \wp_r(d)} (-1)^{I+J} \det\left(B^{(J,I)}\right) \cdot \det\left(A_I^{(J^*,I^*)}\left(K, B^{(J^*,I^*)}\right)\right) = \det\left(B(I \smallsetminus K, A_I)\right).$$

Letting $s = |I \smallsetminus K| < r$, we now take the Laplace expansion of this last determinant along the column set $(I \smallsetminus K)$ and we get

$$\det\left(B(I \smallsetminus K, A_I)\right) = \sum_{J \in \wp_s(d)} (-1)^{J + (I \smallsetminus K)} \det\left(B^{(J, I \smallsetminus K)}\right) \cdot *,$$

where $*$ is the determinant of a certain matrix which is irrelevant for us. It follows from Lemma 6.7 that

$$\det\left(B(I\smallsetminus K, A_I)\right)=0.$$

Hence, we obtain

$$e_I\cdot\theta^*_{K/k,S,T}(0)\cdot w_I = x_T\sum_{J\in\wp_r(d)}(-1)^{I+J}\det\left(B^{(J,I)}\right)\cdot\det\left(A_I^{(J^*,I^*)}\right)\cdot w_I.$$

Furthermore, if we set

$$\zeta=\prod_{i\in I}\zeta_i,$$

then

$$\begin{aligned}
e_I\cdot\theta^*_{K/k,S,T}(0)\cdot w_I &= x_T\cdot\zeta\cdot\sum_{J\in\wp_r(d)}(-1)^{I+J}\det\left(B^{(J,I)}\right)\cdot\det\left(A_I'^{(J^*,I^*)}\right)\cdot w_I\\
&= x_T\cdot\zeta\cdot\sum_{J\in\wp_r(d)}(-1)^{I+J}\det\left(B^{(J,I)}\right)\cdot\wedge^r\tilde{\rho}_I\circ\lambda\circ\psi\circ p(b_J)\\
&= \wedge^r\tilde{\rho}_I\circ\lambda\circ\psi\circ p\left(x_T\cdot\zeta\cdot\sum_{J\in\wp_r(d)}(-1)^{I+J}\det\left(B^{(J,I)}\right)\cdot b_J\right),
\end{aligned}$$

where $b_J=b_{j_1}\wedge\ldots\wedge b_{j_r}$ if $J=(j_1,\ldots,j_r)$.

We let

$$\eta_I'=\sum_{J\in\wp_r(d)}(-1)^{I+J}\det\left(B^{(J,I)}\right)\cdot b_J.$$

Remark that

$$\eta_I'\in\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^r F.$$

Lemma 6.8. *With the same notation as above, we have* $e_I\cdot\eta_I'=\eta_I'$.

Proof. We let $\phi_{J,I}$ be the usual map

$$\phi_{J,I}:\bigoplus_{j\notin J}\mathbb{C}[G]\cdot b_j\xrightarrow{\phi_{res}}\mathbb{C}F\xrightarrow{q}\bigoplus_{i\notin I}\mathbb{C}[G]\cdot b_i,$$

where q is the natural projection map. The matrix $B^{(J,I)}$ corresponds to the morphism $\phi_{J,I}$. We will show that if $\chi\notin\widehat{G}_{r,S,I}$, then $\phi^{\chi}_{J,I}$ is singular. Let us start with $\chi\notin\widehat{G}_{r,S,I}$, then

$$\dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\phi)\cdot e_\chi)>r.$$

Suppose that

$$(\mathbb{C}\mathrm{Ker}(\phi)\cdot e_\chi)\bigcap\left(\bigoplus_{j\notin J}\mathbb{C}[G]\cdot b_j\cdot e_\chi\right)=0,$$

then we would have

$$\begin{aligned}
\dim_{\mathbb{C}}\left(\mathbb{C}\mathrm{Ker}(\phi)\cdot e_\chi+\bigoplus_{j\notin J}\mathbb{C}[G]\cdot b_j\cdot e_\chi\right) &= \dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\phi)\cdot e_\chi)+d-r\\
&> r+d-r\\
&= d.
\end{aligned}$$

This would be a contradiction.

Now, let $\chi\in\widehat{G}_{r,S}\smallsetminus\widehat{G}_{r,S,I}$, then

$$\dim_{\mathbb{C}}(\mathbb{C}\mathrm{Im}(\phi)\cdot e_\chi)=d-r.$$

Suppose for the sake of contradiction that $\phi^{\chi}_{J,I}$ is not singular. Then

$$\mathrm{Ker}(\phi^{\chi}) \bigcap \left(\bigoplus_{j \notin J} \mathbb{C}[G] \cdot b_j \right) \cdot e_{\chi} = 0$$

necessarily. From this, we conclude that

$$\mathrm{Ker}(\phi^{\chi}) + \left(\bigoplus_{j \notin J} \mathbb{C}[G] \cdot b_j \right) \cdot e_{\chi} = \mathbb{C}F \cdot e_{\chi}. \tag{40}$$

Since $\chi \notin \widehat{G}_{r,S,I}$, there exists $i_0 \in I$ such that $G_{i_0} \nsubseteq \mathrm{Ker}(\chi)$. Since $\pi(b_{i_0}) = w_{i_0} - w_0$, it follows that $\pi(e_{\chi} \cdot b_{i_0}) = e_{\chi} \cdot (w_{i_0} - w_0) = 0$. Indeed, $G_0, G_{i_0} \nsubseteq \mathrm{Ker}(\chi)$. (Here is an instance where we use the hypothesis $S \neq S_{min}$.) We conclude that

$$e_{\chi} \cdot b_{i_0} \in \mathrm{Im}(\phi^{\chi}).$$

This means that there exists $x \in \mathbb{C}F \cdot e_{\chi}$ such that $\phi^{\chi}(x) = e_{\chi} \cdot b_{i_0}$. Because of (40), we can write $x = y' + y$ for some $y' \in \mathrm{Ker}(\phi^{\chi})$ and some

$$y \in \left(\bigoplus_{j \notin J} \mathbb{C}[G] \cdot b_j \right) \cdot e_{\chi}.$$

Then, $\phi^{\chi}(x) = \phi^{\chi}(y)$ and since $q(e_{\chi} \cdot b_{i_0}) = 0$, we have

$$\phi^{\chi}_{J,I}(y) = 0.$$

But $y \neq 0$, since $e_{\chi} \cdot b_{i_0} \neq 0$; therefore, $\phi^{\chi}_{J,I}$ is singular as we wanted to show. □

The next step is to show that

$$\eta'_I \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Ker}(\phi),$$

but before doing so, we state here a lemma taken from [6].

Lemma 6.9. *Let V be a finite dimensional vector space over a field D. Let θ be a D-linear endomorphism of V such that $\mathrm{Ker}(\theta) \cap \mathrm{Im}(\theta) = 0$. Then for any natural number m with $m \leq \dim_D(\mathrm{Ker}(\theta))$ there exists an exact sequence of D-modules of the form*

$$0 \longrightarrow \bigwedge_{D}^{m} \mathrm{Ker}(\theta) \longrightarrow \bigwedge_{D}^{m} V \xrightarrow{\Theta} \bigoplus_{t=1}^{m} \bigwedge_{D}^{m} V,$$

where the second arrow is the m-th exterior power of the inclusion $\mathrm{Ker}(\theta) \subseteq V$ and, for each integer j with $1 \leq j \leq m$, the projection Θ_j of Θ to the j-th direct summand $\wedge_D^m V$ of the right hand module is induced by

$$v_1 \wedge \ldots \wedge v_m \mapsto \sum_{K \in \wp_j(m)} \theta_{1,K}(v_1) \wedge \ldots \wedge \theta_{m,K}(v_m),$$

with $\theta_{i,K}$ equal to θ if $i \in K$ and equal to the identity map on V otherwise.

Proof. See Lemma 8.4 of [6]. □

Remark. Lemma 6.9 would not be true without the hypothesis $\mathrm{Ker}(\theta) \cap \mathrm{Im}(\theta) = 0$, as was pointed out by Burns in [6]. This explains the necessity of Lemma 6.4.

Lemma 6.10. *With the same notation as above, we have*

$$\eta'_I \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Ker}(\phi).$$

Proof. The proof is a little tedious, but the idea is to use Lemma 6.9 above. We follow closely the proof of Theorem 8.1(i) in [6]. By Lemma 6.8, we just have to show that

$$e_\chi \cdot \eta_I' \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Ker}(\phi) \cdot e_\chi,$$

for all $\chi \in \widehat{G}_{r,S,I}$. Hence, we will apply Lemma 6.9 with $D = \mathbb{C}$, $V = \mathbb{C}F \cdot e_\chi$, $\theta = \phi^\chi$ and $m = r = \dim_{\mathbb{C}}(\mathbb{C}\mathrm{Ker}(\phi) \cdot e_\chi)$. The notation being as in Lemma 6.9, we have to show

$$w_s := \Theta_s(e_\chi \cdot \eta_I') = \sum_{J \in \wp_r(d)} (-1)^{I+J} \det(B^{(J,I)}) \cdot \Theta_s(e_\chi \cdot b_J) = 0,$$

for all $s = 1, \ldots, r$. Now, a basis for the $\mathbb{C}$-vector space $\wedge_{\mathbb{C}}^r(\mathbb{C}F \cdot e_\chi)$ is given by the elements $e_\chi \cdot b_L$ as L runs over all elements in $\wp_r(d)$. Let

$$\rho_L : \bigwedge_{\mathbb{C}}^{r}(\mathbb{C}F \cdot e_\chi) \longrightarrow \mathbb{C} \cdot b_L \cdot e_\chi$$

be the natural projection. We have to show that for all such L, one has

$$\rho_L(w_s) = 0.$$

Using the definition of Θ_s, we see that that $w_s = \Theta_s(e_\chi \cdot \eta_I')$ is a sum of elements which are scalar multiples of elements of the form

$$v_{J,K} := \left(\bigwedge_{\substack{1 \le a \le r \\ a \notin K}} e_\chi \cdot b_{j_a} \right) \wedge \left(\bigwedge_{\substack{1 \le a \le r \\ a \in K}} \phi^\chi(e_\chi \cdot b_{j_a}) \right),$$

where $K \in \wp_s(r)$ and $J \in \wp_r(d)$. Following Burns, we now claim that if $\rho_L(v_{J,K}) \neq 0$, then the set

$$\Sigma_{J,K} = \{ j_a \in J \mid j_a \in L, a \notin K \} \subseteq J$$

has cardinality $r - s$. Indeed, if $|J \smallsetminus L| > s$, then there exists at least one integer a with both $j_a \notin L$ and $a \notin K$, since $|K| = s$. By definition of $v_{J,K}$, we see that $\rho_L(v_{J,K}) = 0$. Hence, $|J \smallsetminus L| = s - f$ for some integer f with $0 \le f \le s$ if $\rho_L(v_{J,K}) \neq 0$. If $j_a \notin L$, necessarily $a \in K$ by definition of $v_{J,K}$. Since $|J \smallsetminus L| = s - f$ and $|K| = s$, there are $s - (s - f) = f$ elements j_a of $J \cap L$ for which $a \in K$. It follows that there are $|J \cap L| - f$ elements j_a of $J \cap L$ with $a \notin K$. But $|J \cap L| = r - (s - f)$; thus, there are $r - s$ elements j_a of $J \cap L$ with $a \notin K$. In other words, if $\rho_L(v_{J,K}) \neq 0$, then

$$|\Sigma_{J,K}| = r - s,$$

and this proves our claim.

If $\rho_L(v_{J,K}) \neq 0$, and $i \in I \cap L$, then Lemma 6.5 implies that $i = j_a$ for some $a \notin K$. In other words, if $\rho_L(v_{J,K}) \neq 0$, then one necessarily has the inclusion

$$L \cap I \subseteq \Sigma_{J,K}.$$

In summary, when computing $\rho_L(w_s)$, one just has to consider terms which are scalar multiples of terms of the form $v_{J,K}$, where $J \in \wp_r(d)$ and $K \in \wp_s(r)$ are such that $\Sigma_{J,K}$ has cardinality $r - s$ and contains $L \cap I$.

If J is given and $|\Sigma_{J,K}| = r - s$, then K is uniquely determined by $\Sigma_{J,K}$. Indeed, $a \in K$ if and only if $j_a \notin \Sigma_{J,K}$, as a counting argument shows.

Therefore, we have

$$\rho_L(w_s) = \sum_M \rho_L(w_{s,M}),$$

where

(1) The set M runs over the sets of cardinality $r - s$ with $L \cap I \subseteq M \subseteq L$.

(2) The element $w_{s,M}$ is defined as

$$w_{s,M} = \sum_{\substack{J\in\wp_r(d)\\ M\subseteq J}} (-1)^{I+J}\det(B^{(J,I)})\cdot\phi_M^{\chi}(e_\chi\cdot b_{j_1})\wedge\ldots\wedge\phi_M^{\chi}(e_\chi\cdot b_{j_r}),$$

with

$$\phi_M^{\chi}(e_\chi\cdot b_a) = \begin{cases} e_\chi\cdot b_a, & \text{if } a\in M,\\ q(\phi^{\chi}(e_\chi\cdot b_a)), & \text{if } a\notin M,\end{cases}$$

where q is defined as follows.

(3) The map q is the natural projection

$$q:\mathbb{C}F\cdot e_\chi\longrightarrow\bigoplus_{\ell\in L\smallsetminus M}\mathbb{C}e_\chi\cdot b_\ell.$$

Hence, we just have to show that

$$\rho_L(w_{s,M}) = 0,$$

for all s and M as above.

A simple computation shows that

$$\phi_M^{\chi}(e_\chi\cdot b_{j_1})\wedge\ldots\wedge\phi_M^{\chi}(e_\chi\cdot b_{j_r}) = \operatorname{sgn}(\tau_{J,M})\cdot\operatorname{sgn}(\tau_{L,M})\det\left(B^{((J\smallsetminus M)^*,(L\smallsetminus M)^*)}\right)\cdot b_L.$$

Using the equality

$$\operatorname{sgn}(\tau_{[d],J})\cdot\operatorname{sgn}(\tau_{J,M}) = \operatorname{sgn}(\tau_{[d],M})\cdot\operatorname{sgn}(\tau_{[d]\smallsetminus M,J\smallsetminus M}),$$

we obtain that $\rho_L(w_{s,M})$ is equal to

$$\operatorname{sgn}(\tau_{[d],I})\cdot\operatorname{sgn}(\tau_{L,M})\cdot\operatorname{sgn}(\tau_{[d],M})\sum_{\substack{J\in\wp_r(d)\\ M\subseteq J}}\operatorname{sgn}(\tau_{[d]\smallsetminus M,J\smallsetminus M})\det\left(B^{(J,I)}\right)\det\left(B^{((J\smallsetminus M)^*,(L\smallsetminus M)^*)}\right)\cdot b_L.$$

This last sum is parametrized by the elements $J\smallsetminus M$. It follows that this last sum is the Laplace expansion along the first s columns of the $(d-r+s)\times(d-r+s)$-matrix N defined as

$$N = (n_{ij}) = \begin{cases} B_{([d]\smallsetminus M)_i,(L\smallsetminus M)_j}, & \text{if } 1\le j\le s,\\ B_{([d]\smallsetminus M)_i,([d]\smallsetminus I)_{j-s}}, & \text{if } s<j\le d-r+s.\end{cases}$$

Since $L\cap I\subseteq M\subseteq L$, we see that if $\ell\in L\smallsetminus M$, then $\ell\notin I$. It follows that N has two identical columns and $\rho_L(w_{s,M})=0$. This concludes the proof of the lemma. □

Lemma 6.8 implies that

$$\zeta\cdot\eta_I' = \eta_I'.$$

Lemma 6.10 implies that $\wedge^r p(\eta_I') = \eta_I'$ and Lemma 6.8 implies that

$$e_I\cdot\theta^*_{K/k,S,T}(0)\cdot w_I = \wedge^r\lambda\circ\psi\left(x_T\cdot\eta_I'\right),$$

since $\tilde\rho_I$ is the identity on $\mathbb{C}Y_{K,S}\cdot e_I$. The map $\wedge^r\lambda$ being an isomorphism, we deduce that

$$\eta_I = x_T\cdot\wedge^r\psi(\eta_I'). \tag{41}$$

Lemma 6.11. *Let* $\psi_1,\ldots,\psi_r\in\operatorname{Hom}_{\mathbb{Z}[G]}(\operatorname{Ker}(\phi),\mathbb{Z}[G])$, *then*

$$\psi_1\wedge\ldots\wedge\psi_r(\eta_I')\in\mathbb{Z}[G].$$

Proof. Starting with the short exact sequence of $\mathbb{Z}[G]$-modules

$$0\longrightarrow\operatorname{Ker}(\phi)\longrightarrow F\longrightarrow\operatorname{Im}(\phi)\longrightarrow 0,$$

and since $\operatorname{Im}(\phi)$ has no $\mathbb{Z}$-torsion, we conclude that the morphism

$$\operatorname{Hom}_{\mathbb{Z}[G]}(F,\mathbb{Z}[G])\longrightarrow\operatorname{Hom}_{\mathbb{Z}[G]}(\operatorname{Ker}(\phi),\mathbb{Z}[G])$$

is surjective. Let $\bar{\psi}_i$ be lifts of ψ for $i = 1, \ldots, r$. We then have

$$\begin{aligned}\psi_1 \wedge \ldots \wedge \psi_r(\eta'_I) &= \bar{\psi}_1 \wedge \ldots \wedge \bar{\psi}_r(\eta'_I) \\ &= \sum_{J \in \wp_r(d)} (-1)^{J+I} \det\left(B^{(J,I)}\right) \det(\bar{\psi_k}(b_{j_l}))_{k,l} \in \mathbb{Z}[G],\end{aligned}$$

since as noted before $B \in M_d(\mathbb{Z}[G])$. □

From Lemma 6.10, we conclude in particular that

$$\eta'_I \in \left(\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^r \mathrm{Ker}(\phi)\right) \bigcap \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r F\right) = \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r \mathrm{Ker}(\phi),$$

and since $x_T \in \mathbb{Q}[G]$, it follows from (41) that

$$\eta_I \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r E_{K,S,T}.$$

If $\phi_1, \ldots, \phi_r \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_{K,S,T}, \mathbb{Z}[G])$, then

$$\begin{aligned}\phi_1 \wedge \ldots \wedge \phi_r(\eta_I) &= x_T \cdot \phi_1 \wedge \ldots \wedge \phi_r \left(\wedge^r \psi(\eta'_I)\right) \\ &= x_T \cdot (\phi_1 \circ \psi) \wedge (\phi_2 \circ \psi) \wedge \ldots \wedge (\phi_r \circ \psi)(\eta'_I) \in \mathbb{Z}_{(\ell)}[G],\end{aligned}$$

for all $\ell \mid z$, because of Lemma 6.11 combined with the fact that $x_T \in \mathbb{Z}_\ell[G]$ for all $\ell \mid z$. Since z was chosen arbitrarily, we conclude that

$$\eta_I \in \Lambda_{K/k,S,T},$$

and this completes the proof that the equivariant Tamagawa number conjecture implies Conjecture 4.16 under the assumption $Cl_{K,S,T} = 1$.

6.3. **Main theorem.** We are now done proving the main result of this paper:

Theorem 6.12. *The equivariant Tamagawa number conjecture implies Conjecture 4.16.*

Proof. In §6.2, we showed that the equivariant Tamagawa number conjecture implies Conjecture 4.16 under the assumption $Cl_{K,S,T} = 1$. Proposition 5.3 allows us to remove this assumption. □

7. Explicit consequences when the base field is $\mathbb{Q}$

When the base field is $\mathbb{Q}$, the equivariant Tamagawa number conjecture is known to be true:

Theorem 7.1 (Burns-Greither, Flach)**.** *Let $K/\mathbb{Q}$ be a finite abelian extension of number fields, then the equivariant Tamagawa number conjecture is true.*

Proof. The main result of [8] is precisely this result away from the prime 2. In [13] and [14], Flach completed their result at the prime 2. □

As a consequence of our results, we obtain:

Corollary 7.2. *Conjecture 4.16 is true when the base field is $\mathbb{Q}$.*

Proof. This follows from Theorems 6.12 and 7.1. □

From Lemma 4.17 combined with the previous corollary, we obtain:

Corollary 7.3. *The extended abelian Stark conjecture is true when the base field is $\mathbb{Q}$.*

Remark. This corollary includes the (S,T)-version of the extended abelian Stark conjecture, that is the Emmons-Popescu conjecture, and the S-version of the extended abelian rank one Stark conjecture, that is the Erickson-Stark conjecture, as stated in [12] (Conjecture 4.1) or in [28] (Conjecture 3.6).

Corollary 7.4. *Conjecture 4.23 is true when the base field is $\mathbb{Q}$.*

Proof. Indeed, the Rubin-Stark conjecture is true when the base field is $\mathbb{Q}$, because of Theorem 7.1 combined with the previous work of Burns in [6] (in the rank one case, it is also possible to write down explicitly a Stark unit as a cyclotomic unit or a normalized Gauss sum depending on the nature of the split prime). Hence, we can conclude the desired result using Proposition 4.24. □

Moreover, we obtain the following concrete results. We record them here, since we think they are interesting.

We let $K/\mathbb{Q}$ be a finite abelian extension of number fields and S a 1-cover such that $|S| \geq 3$ and $S \neq S_{min}$. As throughout this paper, $S = \{v_0, v_1, \dots, v_n\}$ and $S_{min} = \{v_1, \dots, v_m\}$ for some integer m satisfying $1 \leq m \leq n$. For each $i = 1, \dots, m$, we let $L_i = K^{G_i}$, $\Gamma_i = G/G_i$ and $n_i = |G_i|$. We also let w_i' be a place of L_i lying between v_i and w_i.

In [28], we explained how to go from the (S, T)-version of Conjecture 4.12 to a S-version. In particular, if v_i is the real infinite place, then we have the following theorem.

Corollary 7.5. *If $v_i = \infty \in S_{min}$, the unique infinite place of $\mathbb{Q}$, we have $L_i = K^+$ and $\Gamma_i = G^+ = \mathrm{Gal}(K^+/\mathbb{Q})$. Then there exists $\eta \in E_{K^+,S}$ such that $|\eta|_{w'} = 1$ whenever w' is a place of K^+ not lying above ∞. Moreover,*

$$\theta'_{K^+/k,S}(0) = -\sum_{\gamma \in G^+} \log |\eta^\gamma|_{w_i'} \cdot \gamma^{-1},$$

and the extension $K^+(\sqrt{\eta})/\mathbb{Q}$ is abelian.

Proof. It suffices to remark that $w_{K^+} = |G_\infty| = 2$. □

Remark. In [28] we were able to show the existence of an η satisfying the two first properties of this last corollary, but not the abelian condition. Instead of the equivariant Tamagawa number conjecture, we used a conjecture of Gross (Conjecture 7.6 of [16]) and the Hasse principle for powers.

Corollary 7.6. *If $v_i \in S_{min}$ is a finite prime, then let $R_i = S \smallsetminus \{v_i\}$. One has*

$$\mathrm{Ann}_{\mathbb{Z}[\Gamma_i]}(\mu_{L_i}) \cdot \frac{\theta_{L_i/k,R_i}(0)}{n_i} \subseteq \mathbb{Z}[\Gamma_i],$$

where μ_{L_i} is the finite group of roots of unity in $L_i^\times$.

Proof. This follows from point (1) of Proposition 4.16 and Lemma 2.2 of [28]. □

Remark. In other words, Hypothesis 2.2 of [29] is always satisfied when the base field is $\mathbb{Q}$. In [28] we were able to show this corollary only in the case where $v_i \in S_{min}$ is a finite prime which is unramified in $K/\mathbb{Q}$. Again, instead of using the equivariant Tamagawa number conjecture, we used a conjecture of Gross (Conjecture 7.6 of [16]).

The S-version of Conjecture 4.12 implies that the special value

$$\frac{w_{L_i}\theta_{L_i/k,R_i}(0)}{n_i} \in \mathbb{Z}[\Gamma_i]$$

annihilates a certain subgroup of Cl_{L_i}. Moreover, the principal ideals obtained can be generated by elements satisfying the usual conditions of the Brumer-Stark conjecture (they are anti-units and they satisfy the abelian condition). It would be interesting to study the question of whether or not it annihilates the whole class group Cl_{L_i}. This seems to be related in some particular cases to previous works of Greither-Kučera in [15] and of Burns-Haywards in [9]. For more on these matters, see [29].

8. Conclusion

As a concluding remark, we emphasize that the seemingly simple results of §7 follow from the works of Burns-Greither ([8]) and of Flach ([13] and [14]) in which they use the following results among others which are all known when the base field is $\mathbb{Q}$: the Leopoldt conjecture, the vanishing of the μ-invariant, the main conjecture in Iwasawa theory, and the Euler system of cyclotomic units. These are all deep results and are among central results in number theory, some of which remain to be extended to other base fields than $\mathbb{Q}$. We think that it would be interesting to find a more classical proof of the results of §7 without relying on the equivariant Tamagawa number conjecture.

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Department of Mathematical Sciences, Binghamton University, Binghamton, New York 13902-6000

E-mail address: `vallieres@math.binghamton.edu`