A REMARK ON A RESULT OF IWASAWA ON HECKE CHARACTERS OF TYPE A_0

DANIEL VALLIÈRES

Abstract. Let K and L be two number fields. On page 103 of [2], Iwasawa proved a theorem about Hecke characters of K with values in L under the assumptions that K is Galois over $\mathbb{Q}$, K is a CM-field and $K \subseteq L$. In this paper, we remove these hypotheses.

Contents

1. Introduction

Let K and L be two number fields. There is a natural group morphism which associates to a Hecke character of K with values in L its infinity-type. In [2], Iwasawa proved a theorem about the image of this group morphism in the case where K is a CM-field which is Galois over $\mathbb{Q}$ and $K \subseteq L$. Our goal here is to remove these hypotheses.

This paper is subdivided as follows. After setting up some basic notation in §1.1, we remind the reader in §1.2 about complex conjugations and CM-fields. In §2, we review the theory of idelic Hecke characters. Iwasawa's theorem is formulated in §3 without restrictive hypotheses.

1.1. Basic notation and results.

Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. We will denote $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ more simply by $G_{\mathbb{Q}}$. By a number field, we mean a subfield of $\bar{\mathbb{Q}}$ which is of finite degree over $\mathbb{Q}$.

Let K be a number field. If L is a field of characteristic zero, then we let

$$\Sigma_K(L) := \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K, L),$$

and we write Σ_K instead of $\Sigma_K(\bar{\mathbb{Q}})$. The ring of integers of K will be denoted by O_K and we set $E_K = O_K^{\times}$ to be the group of units of O_K. Moreover, the group of fractional ideals is denoted by I_K. We have a group morphism $\mathbb{N} : I_K \longrightarrow \mathbb{Q}_{>0}$, where $\mathbb{N}$ denotes the absolute norm of a fractional ideal. Given $\lambda \in K^{\times}$, one has $|N_{K/\mathbb{Q}}(\lambda)| = \mathbb{N}((\lambda))$, where $N_{K/\mathbb{Q}} : K^{\times} \longrightarrow \mathbb{Q}^{\times}$ is the usual norm map and (λ) is the principal ideal generated by $\lambda \in K^{\times}$ also denoted by $\lambda \cdot O_K$. We denote the class group of K by $Cl_K = I_K/P_K$, where P_K is the group of principal ideals of K. More generally, one has the

Date: March 10, 2014.

ray class groups which we now recall. A modulus $\mathfrak{m} = \mathfrak{m}_0 \cdot \mathfrak{m}_\infty$ of K consists of a formal product of an integral ideal $\mathfrak{m}_0$ and a set of real embeddings $\mathfrak{m}_\infty$ of K. The set of primes dividing $\mathfrak{m}_0$ will be denoted by $\mathrm{Supp}(\mathfrak{m}_0)$. If $\mathfrak{m}$ is a modulus of K, then $I_{K,\mathfrak{m}}$ denotes the group of fractional ideals of K relatively prime to $\mathfrak{m}_0$. The subgroup $K^\times(\mathfrak{m})$ of $K^\times$ consists of all the elements of $K^\times$ satisfying $\lambda \equiv 1 \mod {}^\times\mathfrak{m}$ meaning

$$\mathrm{ord}_\mathfrak{p}(\lambda - 1) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{m}_0) \text{ for all } \mathfrak{p} \,|\, \mathfrak{m}_0 \text{ and } \psi(\lambda) > 0 \text{ for all } \psi \in \mathfrak{m}_\infty.$$

The set of principal ideals (λ) where $\lambda \in K^\times(\mathfrak{m})$ is denoted by $P_{K,\mathfrak{m}}$; it is a subgroup of $I_{K,\mathfrak{m}}$. The quotient $Cl_{K,\mathfrak{m}} = I_{K,\mathfrak{m}}/P_{K,\mathfrak{m}}$ is the ray class group modulo $\mathfrak{m}$. The units of O_K inside of $K^\times(\mathfrak{m})$ form a subgroup of E_K which is denoted by $E_{K,\mathfrak{m}}$ and its torsion subgroup is denoted by $\mu_{K,\mathfrak{m}}$. If $\mathfrak{m} = O_K$, we recover the usual class group Cl_K, the units E_K and the roots of unity μ_K. It is well-known that both $[E_K : E_{K,\mathfrak{m}}]$ and $Cl_{K,\mathfrak{m}}$ are finite for all moduli $\mathfrak{m}$ of K.

Given a number field K and a place v of K, we let K_v denote its completion at v, O_v its ring of integers inside K_v and U_v will stand for $O_v^\times$. If m is an integer satisfying $m \geq 1$, we let $U_v^{(m)}$ denote the units of the form $1 + O_v\pi_v^m$, where π_v is a uniformizer at v. The group of ideles with its usual topology is denoted by $\mathcal{J}_K$. As customary, we view $K^\times$ inside of $\mathcal{J}_K$ via the diagonal embedding. We have the usual continous group morphism $||\cdot|| : \mathcal{J}_K \longrightarrow \mathbb{R}_{>0}$ given by

$$x \mapsto ||x|| = \prod_v |x_v|_v,$$

where the product is over all places v of K and the absolute values $|\cdot|_v$ are the usual normalized ones so that the product formula holds. Given a real place v of K, we have the signature morphism $\mathrm{sgn}_v : K_v^\times \longrightarrow \{\pm 1\}$, defined for $x \in K_v^\times$ by

$$\mathrm{sgn}_v(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{otherwise.} \end{cases}$$

We then define the continuous group morphism $\mathrm{sgn} : \mathcal{J}_K \longrightarrow \{\pm 1\}$, by the formula

$$x \mapsto \prod_{v \text{ real}} \mathrm{sgn}_v(x_v),$$

where the product is over all real places of K. If $\lambda \in K^\times$, we have $|N_{K/\mathbb{Q}}(\lambda)| = \mathrm{sgn}(\lambda) \cdot N_{K/\mathbb{Q}}(\lambda)$. An idele $x \in \mathcal{J}_K$ can be written uniquely as $x = x_0 \cdot x_\infty$, where x_0 is the finite part of x and x_∞ its infinite part. If $\lambda \in K^\times$, then $|N_{K/\mathbb{Q}}(\lambda)| = ||\lambda_\infty|| = ||\lambda_0||^{-1}$, the second equality being true because of the product formula. Given a modulus $\mathfrak{m}$ of K, we set

$$\mathcal{J}_{K,\mathfrak{m}} = \{x \in \mathcal{J}_K \,|\, \text{If } v|\mathfrak{m}, \text{then } x_v > 0 \text{ if } v \text{ is real, and } x_v \in U_v^{(\mathrm{ord}_v(\mathfrak{m}))} \text{ if } v \text{ is finite}\},$$

and

$$\mathcal{J}'_{K,\mathfrak{m}} = \{x \in \mathcal{J}_{K,\mathfrak{m}} \,|\, x_v \in U_v \text{ for all finite places } v\},$$

so that $\mathcal{J}_{K,\mathfrak{m}} \cap K^\times = K^\times(\mathfrak{m})$ and $\mathcal{J}'_{K,\mathfrak{m}} \cap K^\times = E_{K,\mathfrak{m}}$. Clearly, $\mathcal{J}'_{K,\mathfrak{m}}$ is an open subgroup of $\mathcal{J}_K$ and we remind the reader that any open subgroup of $\mathcal{J}_K$ contains a $\mathcal{J}'_{K,\mathfrak{m}}$ for some modulus $\mathfrak{m}$. The link between ideles and ideals is given by the group morphism $\iota_K : \mathcal{J}_K \longrightarrow I_K$ defined by

$$x \mapsto \iota_K(x) = \prod_v \mathfrak{p}_v^{\mathrm{ord}_v(x_v)},$$

where the product is over all finite places of K and $\mathfrak{p}_v$ is the prime ideal corresponding to v. If $x \in \mathcal{J}_K$, then $\mathbb{N}(\iota_K(x)) = ||x_0||^{-1}$. Given a modulus $\mathfrak{m}$ of K, we clearly have $\iota_K(\mathcal{J}_{K,\mathfrak{m}}) \subseteq I_{K,\mathfrak{m}}$, and this last inclusion induces an isomorphism of groups

$$\mathcal{J}_{K,\mathfrak{m}}/K^\times(\mathfrak{m})\mathcal{J}'_{K,\mathfrak{m}} \xrightarrow{\simeq} Cl_{K,\mathfrak{m}}.$$

At last, we state the following simple lemma whose proof is left to the reader.

Lemma 1.1. *Let G be an abelian group and let H be a subgroup such that G/H is finite. Let A be an abelian group without $\mathbb{Z}$-torsion. Suppose we are given two group morphims $f_i : G \longrightarrow A$ for $i = 1, 2$, satisfying $f_1(h) = f_2(h)$ for all $h \in H$. Then $f_1(g) = f_2(g)$ for all $g \in G$.*

1.2. **Complex conjugations and CM-fields.** The CM-fields play a prominent role in the theory of Hecke characters, so we recall here some of their properties. We refer to §2 of [4] for omitted proofs.

A CM-field is a totally complex number field K which is a quadratic extension of a totally real field, usually denoted by K^+. Throughout this paper, we let $\rho \in \mathrm{Aut}(\mathbb{C})$ denote the complex conjugation field automorphism of $\mathbb{C}$. If K is a number field, then an automorphism $j \in \mathrm{Aut}_{\mathbb{Q}}(K)$ is called a complex conjugation of K if there exists a complex embedding $\psi \in \Sigma_K(\mathbb{C})$ such that $\psi \circ j = \rho \circ \psi$. A complex conjugation depends only on the place v corresponding to ψ and so might be denoted sometime by j_v. Note that a complex conjugation j necessarily satisfies $j^2 = 1$. Moreover, a complex conjugation is never trivial. We also remark that one can talk about complex conjugations of $\bar{\mathbb{Q}}$ is a similar way as we just did for number fields. If K/k is a finite Galois extension of number fields such that k is totally real, then every complex embedding of K induces a complex conjugation. In particular, if K is a CM-field, then it is a Galois extension of K^+ and every complex embedding induces a complex conjugation which leads us to the following characterization of CM-fields.

Proposition 1.2. *A totally complex field K is a CM-field if and only if the following two conditions are satisfied:*

(1) *Every complex embedding $\psi \in \Sigma_K(\mathbb{C})$ induces a complex conjugation.*
(2) *The complex conjugations j_v do not depend on the places v of K.*

If K is a number field containing a CM-field, then it necessarily contains a maximal CM-field, denoted by K_{CM}, because of the following result.

Proposition 1.3. *The compositum of CM-fields is again a CM-field.*

2. Hecke characters

Our main sources for this section consist of [7], and also [1], page 208 and the following.

2.1. **Infinity-types.** We shall denote the free abelian group on $\Sigma_K = \Sigma_K(\bar{\mathbb{Q}})$ by Z_K. Given

$$\gamma = \sum_{\tau \in \Sigma_K} n_\tau \cdot \tau \in Z_K,$$

for some $n_\tau \in \mathbb{Z}$, it induces a group morphism $N_\gamma : K^\times \longrightarrow \bar{\mathbb{Q}}^\times$ defined by the formula

$$\lambda \mapsto N_\gamma(\lambda) := \prod_{\tau \in \Sigma_K} \tau(\lambda)^{n_\tau}.$$

We will also use the notation $\lambda^\gamma = N_\gamma(\lambda)$. We get a group morphism

$$Z_K \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K^\times, \bar{\mathbb{Q}}^\times). \tag{1}$$

We remark that $G_{\mathbb{Q}}$ acts naturally (on the left) on both Z_K and $\mathrm{Hom}_{\mathbb{Z}}(K^\times, \bar{\mathbb{Q}}^\times)$ via the formulas $g \cdot \tau := g \circ \tau$, whenever $\tau \in \Sigma_K$ and $g \cdot f := g \circ f$, whenever $f \in \mathrm{Hom}_{\mathbb{Z}}(K^\times, \bar{\mathbb{Q}}^\times)$. The group morphism (1) is $G_{\mathbb{Q}}$-equivariant. The next lemma will be useful for proving Proposition 2.2 below.

Lemma 2.1. *Let N be the Galois closure of K in $\bar{\mathbb{Q}}$, $G = \mathrm{Gal}(N/\mathbb{Q})$ and $H = \mathrm{Gal}(N/K)$. For each $\tau \in \Sigma_K(N)$, let $\tilde{\tau} \in G$ be a lift of τ to N. We fix a $\tau_0 \in \Sigma_K(N)$ and a normalized non-archimedean valuation v of N. If $G_v \subseteq \tilde{\tau}_0 H \tilde{\tau}_0^{-1}$, then $v \circ \tau \neq v \circ \tau_0$ for all $\tau \in \Sigma_K(N)$ satisfying $\tau \neq \tau_0$.*

Proof. If $v \circ \tau = v \circ \tau_0$, then $\tilde{\tau}^{-1} \cdot v|_K = \tilde{\tau}_0^{-1} \cdot v|_K$. Hence, there exists $h \in H$ such that $h \cdot \tilde{\tau}^{-1} \cdot v = \tilde{\tau}_0^{-1} \cdot v$. This implies $\tilde{\tau}_0 h \tilde{\tau}^{-1} \in G_v \subseteq \tilde{\tau}_0 H \tilde{\tau}_0^{-1}$. But then, we get $\tilde{\tau}^{-1}\tilde{\tau}_0 \in H$ and we deduce that $\tau = \tau_0$. ☐

Proposition 2.2. *The group morphism (1) is injective.*

Proof. We use the same notation as in Lemma 2.1. Let $\gamma = \sum_{\tau \in \Sigma_K} n_\tau \cdot \tau \in Z_K$ be such that $\lambda^\gamma = 1$ for all $\lambda \in K^\times$ and let us fix $\tau_0 \in \Sigma_K$. By Chebotarev's theorem, we can find a non-archimedean place v of N satisfying $G_v \subseteq \tilde{\tau}_0 H \tilde{\tau}_0^{-1}$. Now applying Lemma 2.1, we see that $v \circ \tau \neq v \circ \tau_0$ if $\tau \neq \tau_0$, where v also denotes the normalized valuation of N corresponding to the place v. Let us denote by S the set of valuations of K of the form $v \circ \tau$ for some $\tau \in \Sigma_K$. Then, for all $\lambda \in K^\times$, we have $v(\lambda^\gamma) = 0$ and therefore

$$\sum_{w \in S} w(\lambda) \sum_{\substack{\tau \in \Sigma_K \\ v \circ \tau = w}} n_\tau = 0.$$

Letting $w_0 = v \circ \tau_0$, then by the approximation theorem, we can find $\lambda \in K^\times$ satisfying

(1) $w(\lambda) = 0$ if $w \in S$ and $w \neq w_0$,
(2) $w_0(\lambda) \neq 0$.

For this choice of λ, we deduce that $n_{\tau_0} = 0$. Since τ_0 was arbitrarily chosen, we are done. □

We will need the following definition.

Definition 2.3. Let K and L be two number fields. We define

$$Z_K(L) = \{\gamma \in Z_K \mid \lambda^\gamma \in L^\times \text{ for all } \lambda \in K^\times\}.$$

Remark. Given $\gamma \in Z_K$, it follows from Proposition 2.2 that $\gamma \in Z_K(L)$ if and only if $g \cdot \gamma = \gamma$, for all $g \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$.

Our next goal is to extend the group morphism $N_\gamma : K^\times \longrightarrow L^\times$ to fractional ideals.

Theorem 2.4. *Let $\gamma \in Z_K(L)$, then there is a unique group morphism $\phi_\gamma : I_K \longrightarrow I_L$ such that the following diagram*

$$\begin{array}{ccc} K^\times & \xrightarrow{N_\gamma} & L^\times \\ \downarrow & & \downarrow \\ I_K & \xrightarrow{\phi_\gamma} & I_L \end{array}$$

commutes, where the vertical arrows are the natural group morphisms mapping a non-zero element λ of a number field to its principal ideal (λ).

Proof. Let N be the smallest Galois extension of $\mathbb{Q}$ in $\bar{\mathbb{Q}}$ which contains both K and L. If $\tau \in \Sigma_K$, then we let $\tilde{\tau} \in \mathrm{Gal}(N/\mathbb{Q})$ be a lift of τ to N and we set $\tilde{\gamma} = \sum_{\tau \in \Sigma_K} n_\tau \tilde{\tau} \in \Sigma_N$. We define $\phi_\gamma : I_K \longrightarrow I_N$ by the formula

$$\mathfrak{a} \mapsto \phi_\gamma(\mathfrak{a}) = \prod_{\tau \in \Sigma_K} \tilde{\tau}(\mathfrak{a} \cdot O_N)^{n_\tau}.$$

We remark that ϕ_γ is a group morphism which does not depend on the choice of the lifts $\tilde{\tau}$.

Now, we have the usual injective group morphism $I_L \hookrightarrow I_N$ given by extension of ideals and we will identify I_L with its image in I_N. Our next goal is to show that $\phi_\gamma(I_K) \subseteq I_L$. We remark first that if $\lambda \in K^\times$, then $\phi_\gamma(\lambda \cdot O_K) = \lambda^\gamma \cdot O_N$. Hence, $\phi_\gamma(P_K) \subseteq I_L$.

It is not true in general that $(I_N)^{\mathrm{Gal}(N/L)} = I_L$, but if we choose an integer m divisible by all the ramified primes in N, then

$$(I_{N,mO_N})^{\mathrm{Gal}(N/L)} = I_{L,mO_L}. \tag{2}$$

If $\mathfrak{a} \in I_{K,mO_K}$, then there exists $n \geq 1$ such that $\mathfrak{a}^n = \alpha \cdot O_K$ for some $\alpha \in K^\times$. We then have

$$\phi_\gamma(\mathfrak{a})^n = \phi_\gamma(\alpha \cdot O_K) = \alpha^\gamma \cdot O_N \subseteq I_{L,mO_L}.$$

Since $\phi_\gamma(\mathfrak{a}) \in I_{N,mO_N}$, it follows from (2) and the fact that I_N is without $\mathbb{Z}$-torsion, that $\phi_\gamma(\mathfrak{a}) \in I_{L,mO_L}$; hence, $\phi_\gamma(I_{K,mO_K}) \subseteq I_L$. If $\mathfrak{a} \in I_K$, then $\lambda\mathfrak{a} \in I_{K,mO_K}$ for some $\lambda \in K^\times$ and we have $\phi_\gamma(\lambda\mathfrak{a}) = \lambda^\gamma \phi_\gamma(\mathfrak{a}) \in I_L$. Therefore, $\phi_\gamma(I_K) \subseteq I_L$. The uniqueness part follows from Lemma 1.1. □

Remark. Let us suppose that we are in the situation where $L \subseteq K$ and K is Galois over L with Galois group H. Viewing H as a subset of Σ_K and setting $s_H = \sum_{h \in H} h \in Z_K$, the map $N_{s_H} : K^\times \longrightarrow L^\times$ is just the usual norm map $N_{K/L}$ and ϕ_{s_H} is the usual norm of a fractional ideal of K down to L. In this sense, the group morphisms ϕ_γ can be viewed as generalizations of the relative norm of an ideal.

We note that we now have a group morphism

$$Z_K(L) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(I_K, I_L), \tag{3}$$

given by $\gamma \mapsto \phi_\gamma \in \mathrm{Hom}_{\mathbb{Z}}(I_K, I_L)$.

2.2. **Idelic Hecke characters.** Let us now endow $\bar{\mathbb{Q}}^\times$ with the discrete topology.

Definition 2.5. An idelic Hecke character of K is a continuous group morphism $\mu : \mathcal{J}_K \longrightarrow \bar{\mathbb{Q}}^\times$, for which there exists $\gamma \in Z_K$ satisfying $\mu(\lambda) = \lambda^\gamma$ whenever $\lambda \in K^\times$.

Because of Proposition 2.2, there is a unique $\gamma \in Z_K$ attached to an idelic Hecke character. It is called the infinity-type of μ and will be denoted by γ_μ. Since μ is continuous, $\ker(\mu)$ is an open subgroup of $\mathcal{J}_K$ and therefore contains a subgroup $\mathcal{J}'_{K,\mathfrak{m}}$ for some modulus $\mathfrak{m}$ of K.

Definition 2.6. Let μ be an idelic Hecke character of K. Any modulus $\mathfrak{m}$ of K satisfying $\mathcal{J}'_{K,\mathfrak{m}} \subseteq \ker(\mu)$, is called a modulus of definition for μ. The smallest one (with respect to divisibility of moduli) is called the conductor of μ and is denoted by $\mathfrak{f}$.

The set of idelic Hecke characters H_K of K is a group and we have a group morphism $\delta : H_K \longrightarrow Z_K$ which maps an idelic Hecke character to its infinity-type. The kernel of this group morphism consists of idelic Hecke characters $\mu \in H_K$ satisfying $\gamma_\mu = 0$; hence, they correspond via class field theory to characters of Galois groups of finite abelian extensions of K. An example of an idelic Hecke character of K which is not in the kernel of δ is provided by the norm idelic Hecke character $\mu_{\mathbb{N}} : \mathcal{J}_K \longrightarrow \bar{\mathbb{Q}}^\times$, defined by $x \mapsto \mu_{\mathbb{N}}(x) := \mathrm{sgn}(x) \cdot ||x_0||^{-1}$, where x_0 is the finite part of the idele x. Its infinity-type is given by $T_K = \sum_{\tau \in \Sigma_K} \tau \in Z_K$. Indeed, given any $\lambda \in K^\times$, one has

$$\mu_{\mathbb{N}}(\lambda) = \mathrm{sgn}(\lambda) \cdot ||\lambda_0||^{-1} = \mathrm{sgn}(\lambda) \cdot |N_{K/\mathbb{Q}}(\lambda)| = N_{K/\mathbb{Q}}(\lambda) = \lambda^{T_K}.$$

2.3. **The image of δ.** Let K be a number field. The infinity-type of an idelic Hecke character of the form $\mu_{\mathbb{N}}^a$, where $a \in \mathbb{Z}$, is given by $a \cdot T_K$. Hence, we always have $\mathrm{Im}(\delta) \supseteq \mathbb{Z} \cdot T_K$.

Theorem 2.7. *Let K be a number field and let N be its Galois closure inside $\bar{\mathbb{Q}}$.*

1. *If K is not totally complex, then $\mathrm{Im}(\delta) = \mathbb{Z} \cdot T_K$.*
2. *If K is totally complex and $\gamma \in Z_K$, then $\gamma \in \mathrm{Im}(\delta)$ if and only if there exists an integer a such that $(1 + j)\gamma = aT_K$ for all complex conjugations $j \in \mathrm{Gal}(N/\mathbb{Q})$.*

Proof. The main ingredients in the proof are the Dirichlet unit theorem and the finiteness of $Cl_{K,\mathfrak{m}}$ and $[E_K, E_{K,\mathfrak{m}}]$. The proof is well-known and we refer to [7] for more details. □

We can now recall the notion of weight of an idelic Hecke character.

Definition 2.8. Let μ be an idelic Hecke character of K with infinity-type γ. It follows from Theorem 2.7, that there exists a unique integer ω satisfying $(1 + j)\gamma = \omega T_K$, for all complex conjugations $j \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The integer ω is called the weight of μ (or of γ).

For example, the weight of the norm idelic Hecke character $\mu_{\mathbb{N}}$ is 2. Let us now suppose that M is another number field and that $M \subseteq K$. We have the restriction map $\mathrm{res}_{K/M} : Z_K \longrightarrow Z_M$, defined by

$$\tau \mapsto \mathrm{res}_{K/M}(\tau) = \tau|_M,$$

whenever $\tau \in \Sigma_K$. It is a group morphism which is $G_{\mathbb{Q}}$-equivariant. We also introduce the corestriction map $\mathrm{cor}_{K/M} : Z_M \longrightarrow Z_K$, defined by

$$\sigma \mapsto \mathrm{cor}_{K/M}(\sigma) = \sum_{\substack{\tau \in \Sigma_K \\ \tau|_M = \sigma}} \tau,$$

whenever $\sigma \in \Sigma_M$, which is also a $G_{\mathbb{Q}}$-equivariant group morphism.

Now, still assuming that K and M are both number fields and that $M \subseteq K$, we have the usual continuous injective group morphism $i_{K/M} : \mathcal{J}_M \hookrightarrow \mathcal{J}_K$. If $\mu \in H_K$, then $\mu \circ i_{K/M} \in H_M$ and

$\gamma_{\mu\circ i_{K/M}} = \mathrm{res}_{K/M}(\gamma_\mu)$. We also have the norm map at the level of ideles which is also a continuous group morphism $N_{K/M} : \mathcal{J}_K \longrightarrow \mathcal{J}_M$. If $\mu \in H_M$, then $\mu \circ N_{K/M} \in H_K$ and $\gamma_{\mu\circ N_{K/M}} = \mathrm{cor}_{K/M}(\gamma_\mu)$.

If K is a totally complex number field, then its Galois closure N may or may not contain a CM-field. In the former case, we let N' be the maximal CM-subfield of N and in the latter case, we let N' be the maximal real subfield. By a maximality argument, the extension $N'/\mathbb{Q}$ is Galois. We also let $M = K \cap N'$ and if $\sigma \in \Sigma_M$, we let $\mathrm{Hom}_\sigma(K, N)$ be the set of $\tau \in \Sigma_K(N)$ satisfying $\tau|_M = \sigma$.

Proposition 2.9. *The notation being as above, given any $\sigma \in \Sigma_M$, the group $\mathrm{Gal}(N/N')$ acts transitively on $\mathrm{Hom}_\sigma(K, N)$.*

Proof. The proof is a simple exercise in Galois theory and is left to the reader. □

We keep the same notation as above.

Lemma 2.10. *Let K be a totally complex number field and let $\gamma \in \mathrm{Im}(\delta)$, then $\gamma \in Z_K(N')$.*

Proof. Indeed, from Theorem 2.7, there exists $a \in \mathbb{Z}$ such that $(1 + j)\gamma = aT_K$, for all complex conjugations $j \in \mathrm{Gal}(N/\mathbb{Q})$. Hence, if j_1 and j_2 are two complex conjugations of N, then

$$j_1 j_2 \gamma = j_1(aT_K - \gamma) = aT_K - (aT_K - \gamma) = \gamma.$$

It follows from the remark following Definition 2.3 that $\gamma \in Z_K(N')$, since $\mathrm{Gal}(N/N')$ is generated by the elements of the form $j_1 \cdot j_2$, where j_1 and j_2 run over all complex conjugations of N. □

Corollary 2.11. *Let K be a totally complex number field and as above, let $M = K \cap N'$. If*

$$\gamma = \sum_{\tau\in\Sigma_K} n_\tau \cdot \tau \in \mathrm{Im}(\delta),$$

then $n_\tau = n_{\tau'}$, whenever $\tau|_M = \tau'|_M$.

Proof. This is a consequence of Proposition 2.9 and Lemma 2.10. □

Because of this last corollary, given $\gamma = \sum_{\tau\in\Sigma_K} n_\tau \cdot \tau \in \mathrm{Im}(\delta)$, we can define

$$\gamma_1 = \sum_{\sigma\in\Sigma_M} m_\sigma \cdot \sigma \in Z_M,$$

by setting $m_\sigma = n_\tau$ if $\tau|_M = \sigma$.

Proposition 2.12. *With the notation as above, we have*

1. $\mathrm{cor}_{K/M}(\gamma_1) = \gamma$,
2. $\mathrm{res}_{K/M}(\gamma) = [K : M] \cdot \gamma_1$,
3. *If ω is the weight of γ, then $(1 + j)\gamma_1 = \omega T_M$ for all complex conjugations $j \in \mathrm{Gal}(N/\mathbb{Q})$.*

Proof. Points (1) and (2) are clear. Point (3) follows from point (2). □

The following corollary is due to Weil. See [7], page 4.

Corollary 2.13 (Weil)**.** *Let K be a number field.*

1. *If K does not contain a CM-field, then every idelic Hecke character of K is of the form $\mu_0 \cdot \mu_{\mathbb{N}}^a$, where $a \in \mathbb{Z}$ and μ_0 is an idelic Hecke character having trivial infinity-type.*
2. *If K contains a CM-field, then every idelic Hecke character of K is of the form $\mu_0 \cdot \mu_1 \circ N_{K/K_{CM}}$, where μ_1 is an idelic Hecke character of K_{CM} and μ_0 is an idelic Hecke character of K having trivial infinity-type.*

Proof. Using the same notation as in Proposition 2.12, if K is totally complex, but K does not contain any CM-field, then $M = K \cap N'$ is real. It follows from point (3) of Proposition 2.12, that $(1 + j)\gamma_1 = 2\gamma_1 = \omega T_M$. Hence, ω is even and $\gamma_1 = \frac{\omega}{2} T_M$. Therefore,

$$\gamma = \mathrm{cor}_{K/M}\left(\frac{\omega}{2} \cdot T_M\right) = \frac{\omega}{2} \cdot T_K.$$

This last argument combined with Theorem 2.7 shows that if K is any number field not containing a CM-field, then $\mathrm{Im}(\delta) = \mathbb{Z} \cdot T_K$. Then, if $\mu \in H_K$, one has $\gamma_\mu = aT_K$ for some $a \in \mathbb{Z}$ and thus $\mu \cdot \mu_{\mathbb{N}}^{-a}$ is an idelic Hecke character of K with trivial infinity-type as we wanted to show.

As for point (2), we proceed as follows. Let K be a number field which contains a CM-field and let $\mu \in H_K$ with infinity-type γ. Then the field $M = K \cap N'$ of Proposition 2.12 is K_{CM}, the maximal CM-subfield of K. It follows from point (3) of Proposition 2.12 and Theorem 2.7 that there exists $\mu_1 \in H_M$ such that $\gamma_{\mu_1} = \gamma_1$. On the other hand, we have $\gamma_{\mu_1 \circ N_{K/M}} = \mathrm{cor}_{K/M}(\gamma_1) = \gamma$ by point (1) of Proposition 2.12. Hence, $\mu \cdot (\mu_1 \circ N_{K/M})^{-1}$ is an idelic Hecke character of K with trivial infinity-type as we wanted to show. □

2.4. **Idelic Hecke characters with values in a number field.** We begin this section with the following proposition.

Proposition 2.14. *An idelic Hecke character μ of K takes values in a number field.*

Proof. This is because the ray class groups are finite. We leave the details to the reader. □

This leads us to the following definition.

Definition 2.15. An idelic Hecke character $\mu \in H_K$ is said to take values in a number field L if $\mu(\mathcal{J}_K) \subseteq L$. We let $H_K(L)$ denote the subgroup of H_K consisting of idelic Hecke characters taking values in the number field L.

For example, the norm idelic Hecke character $\mu_{\mathbb{N}}$ is in $H_K(\mathbb{Q})$. By restricting δ to $H_K(L)$, we obtain a group morphism

$$\delta_L : H_K(L) \longrightarrow Z_K(L).$$

We clearly have $\mathrm{Im}\,(\delta_L) \subseteq \mathrm{Im}(\delta) \cap Z_K(L)$ and we will see in §2.6 a characterization of those $\gamma \in \mathrm{Im}(\delta) \cap Z_K(L)$ which are in $\mathrm{Im}(\delta_L)$.

Theorem 2.16. *Let $\mu \in H_K(L)$ and suppose it has infinity-type γ and weight ω. Let $\psi \in \Sigma_L(\mathbb{C})$. Then, for all $x \in \mathcal{J}_K$, we have*

$$|\psi(\mu(x))|^2 = ||x_0||^{-\omega},$$

where x_0 is the finite part of the idele x and $|\cdot|$ is the usual absolute value of $\mathbb{C}$.

Proof. First of all, if μ_0 is an idelic Hecke character of K with trivial infinity-type, then $\mu_0(x)$ is a root of unity. Hence, the theorem is clear if K does not contain any CM-field by Corollary 2.13.

If K contains a CM-field, then let $\mathfrak{m}$ be a modulus of definition for μ and let us suppose first that $x \in \mathcal{J}_{K,\mathfrak{m}}$. Using the fact that the group $\mathcal{J}_{K,\mathfrak{m}}/K^\times(\mathfrak{m})\mathcal{J}'_{K,\mathfrak{m}}$ is finite, a simple computation shows that $|\psi(\mu(x))|^2 = ||x_0||^{-\omega}$. If $x \in \mathcal{J}_K$, then there exists $\lambda \in K^\times$ satisfying $\lambda x \in \mathcal{J}_{K,\mathfrak{m}}$. Hence,

$$|\psi(\mu(x))|^2 = |\psi(\mu(\lambda x)\mu(\lambda^{-1}))|^2 = ||\lambda_0 x_0||^{-\omega} \cdot |\psi(\lambda^{-\gamma})|^2 = ||x_0||^{-\omega},$$

since $|\psi(\lambda^{-\gamma})|^2 = ||\lambda_0||^{\omega}$. □

Now, given $\mu \in H_K$, we let $\mathbb{Q}(\mu)$ be the smallest number field containing $\mu(\mathcal{J}_K)$.

Corollary 2.17. *Let μ be an idelic Hecke character. If $\mathbb{Q}(\mu) \neq \mathbb{Q}$, then $\mathbb{Q}(\mu)$ is a CM-field.*

Proof. We start by showing that $\mathbb{Q}(\mu)$ is necessarily totally complex provided $\mathbb{Q}(\mu) \neq \mathbb{Q}$. Let us suppose that $\psi \in \Sigma_{\mathbb{Q}(\mu)}(\mathbb{R})$ is a real embedding. It follows from Theorem 2.16 that

$$\psi(\mu(x))^2 = ||x_0||^{-\omega} = \mathbb{N}(\iota_K(x))^{\omega},$$

for all $x \in \mathcal{J}_K$. Since $\mathbb{Q}(\mu) \neq \mathbb{Q}$, we necessarily have that ω is odd. If $\mathfrak{p}$ is a prime ideal of K of absolute inertia degree 1 and $x \in \mathcal{J}_K$ is such that $\iota_K(x) = \mathfrak{p}$, then we have $\mu(x) = \pm p^{\omega/2}$, where p is the rational prime of $\mathbb{Q}$ lying below $\mathfrak{p}$. This is impossible, since there are infinitely many such primes $\mathfrak{p}$, but the degree of $\mathbb{Q}(\mu)$ over $\mathbb{Q}$ is finite. Therefore, if $\mathbb{Q}(\mu) \neq \mathbb{Q}$, then $\mathbb{Q}(\mu)$ is totally complex.

Now, let $\psi \in \Sigma_{\mathbb{Q}(\mu)}(\mathbb{C})$ be a complex embedding. From Theorem 2.16, we have

$$\rho\psi(\mu(x)) = \psi\left(\frac{\mathbb{N}(\iota_K(x))^{\omega}}{\mu(x)}\right),$$

for all $x \in \mathcal{J}_K$. Thus, ψ induces a complex conjugation and moreover, this complex conjugation does not depend on ψ. Hence, $\mathbb{Q}(\mu)$ is a *CM*-field by Proposition 1.2. □

2.5. **Hecke characters from the ideal point of view.** The ideal version of an idelic character is given by the following definition.

Definition 2.18. Let $\mathfrak{m}$ be a modulus of K and let $\gamma \in Z_K(L)$. A Hecke character of infinity-type γ with values in L is a group morphism $\chi : I_{K,\mathfrak{m}} \longrightarrow L^{\times}$, such that whenever $\lambda \in K^{\times}$ is such that $\lambda \equiv 1 \mod {}^{\times}\mathfrak{m}$, one has $\chi((\lambda)) = \lambda^{\gamma}$. One says that $\mathfrak{m}$ is a modulus of definition for χ.

Hecke characters are just another way of viewing idelic Hecke characters.

Theorem 2.19. *Let K and L be two number fields. There is a one-to-one correspondence between triples $(\chi, \gamma, \mathfrak{m})$ and $(\mu, \gamma, \mathfrak{m})$, where χ is a Hecke character of K with values in L, infinity-type γ and modulus of definition $\mathfrak{m}$ and $\mu : \mathcal{J}_K \longrightarrow L^{\times}$ is an idelic Hecke character with infinity-type γ and modulus of definition $\mathfrak{m}$.*

Proof. Given a triple $(\chi, \gamma, \mathfrak{m})$ and $x \in \mathcal{J}_K$, one can find $\lambda \in K^{\times}$ satisfying $\lambda x \in \mathcal{J}_{K,\mathfrak{m}}$. The group morphism μ given by $\mu(x) := \chi(\iota_K(\lambda x))\lambda^{-\gamma}$ is the corresponding idelic Hecke character.

Starting with a triple $(\mu, \gamma, \mathfrak{m})$, if $\mathfrak{a} \in I_{K,\mathfrak{m}}$, then let $x \in \mathcal{J}_{K,\mathfrak{m}}$ be such that $\iota_K(x) = \mathfrak{a}$. The corresponding Hecke character $\chi : I_{K,\mathfrak{m}} \longrightarrow L^{\times}$ is given by the formula $\chi(\mathfrak{a}) = \mu(x)$. The details are left to the reader. □

2.6. **The image of δ_L.** If K does not contain a *CM*-field, then $\mathrm{Im}(\delta) \cap Z_K(L) = \mathbb{Z} \cdot T_K = \mathrm{Im}(\delta_L)$, and if $L = \mathbb{Q}$, then $\mathrm{Im}(\delta) \cap Z_K(\mathbb{Q}) = \mathbb{Z} \cdot T_K = \mathrm{Im}(\delta_{\mathbb{Q}})$. Hence, because of Corollaries 2.13 and 2.17, the only interesting cases are when both K and L contain a *CM*-field.

Throughout §2.6, we assume that both K and L are number fields which contain a CM-field.

Theorem 2.20. *Let $\mu \in H_K(L)$ be an idelic Hecke character with infinity-type γ, weight ω and conductor $\mathfrak{f}$. If $\mathfrak{a} \in I_{K,\mathfrak{f}}$ and $x \in \mathcal{J}_{K,\mathfrak{f}}$ is such that $\iota_K(x) = \mathfrak{a}$, then $\mu(x)$ is independent of the choice of $x \in \mathcal{J}_{K,\mathfrak{f}}$. Moreover, one has*

(1) $\phi_{\gamma}(\mathfrak{a}) = \mu(x) \cdot O_L$,
(2) $|\mu(x)|_v = \mathbb{N}(\mathfrak{a})^{\omega}$ *for all infinite places v of L.*

Proof. One has a well-defined group morphism $f_{\gamma} : I_{K,\mathfrak{f}} \longrightarrow I_L$ defined by $\mathfrak{a} \mapsto \mu(x) \cdot O_L$, where $x \in \mathcal{J}_{K,\mathfrak{f}}$ satisfies $\iota_K(x) = \mathfrak{a}$. Now, remark that if $\mathfrak{a} = (\alpha)$ for some $\alpha \in K^{\times}(\mathfrak{f})$, then $f_{\gamma}(\mathfrak{a}) = \phi_{\gamma}(\mathfrak{a})$. Since the ray class group $Cl_{K,\mathfrak{f}}$ is finite and I_L is without $\mathbb{Z}$-torsion, we can apply Lemma 1.1, and we deduce that $\phi_{\gamma}(\mathfrak{a}) = \mu(x) \cdot O_L$, for all $\mathfrak{a} \in I_{K,\mathfrak{f}}$, where $x \in \mathcal{J}_{K,\mathfrak{f}}$ satisfies $\iota_K(x) = \mathfrak{a}$. This ends the proof of point (1). Point (2) is a consequence of Theorem 2.16. □

Corollary 2.21. *If $\mu \in H_K(L)$ is an idelic Hecke character with infinity-type γ and weight ω, then for all fractional ideal $\mathfrak{a} \in I_K$, there exists $\varepsilon(\mathfrak{a}) \in L^{\times}$ such that*

(1) $\phi_{\gamma}(\mathfrak{a}) = \varepsilon(\mathfrak{a}) \cdot O_L$,
(2) $|\varepsilon(\mathfrak{a})|_v = \mathbb{N}(\mathfrak{a})^{\omega}$, *for all infinite places v of L.*

Proof. Starting with $\mathfrak{a} \in I_K$, there exists $\lambda \in K^{\times}$ such that $\lambda\mathfrak{a} \in I_{K,\mathfrak{f}}$, where $\mathfrak{f}$ is the conductor of μ as usual. Letting $x \in \mathcal{J}_{K,\mathfrak{f}}$ be such that $\iota_K(x) = \lambda\mathfrak{a}$, one can check that $\varepsilon(\mathfrak{a}) = \lambda^{-\gamma} \cdot \mu(x) \in L^{\times}$ satisfies both (1) and (2). □

We can now give a characterization of the elements of $\mathrm{Im}(\delta) \cap Z_K(L)$ which are in $\mathrm{Im}(\delta_L)$. We saw the main argument of the proof in slightly different contexts in [5] and [8], but we do not know who first came up with this argument using the m-th power residue symbol.

Theorem 2.22. *Let $\gamma \in \mathrm{Im}(\delta) \cap Z_K(L)$ and suppose that its weight is ω. Then, $\gamma \in \mathrm{Im}(\delta_L)$ if and only if given any fractional ideal $\mathfrak{a}$ of K, there exists $\varepsilon(\mathfrak{a}) \in L^\times$ satisfying*

(1) $\phi_\gamma(\mathfrak{a}) = \varepsilon(\mathfrak{a}) \cdot O_L$,
(2) $|\varepsilon(\mathfrak{a})|_v = \mathbb{N}(\mathfrak{a})^\omega$, *for all infinite places v of L.*

Proof. One direction is precisely Corollary 2.21. Conversely, let us suppose that points (1) and (2) are true. Choose an unramified prime ideal $\mathfrak{p}$ of L which is of absolute inertia degree 1, is relatively prime to w_L and satisfies $\left(\frac{\mathbb{N}(\mathfrak{p})-1}{w_L}, w_L\right) = 1$. This is possible because of Lemma 2.23 below. Let $a, b \in \mathbb{Z}$ be such that

$$a \cdot \frac{\mathbb{N}(\mathfrak{p}) - 1}{w_L} + b \cdot w_L = 1,$$

and let us define $\mathfrak{m} = p \cdot O_K$, where p is the prime of $\mathbb{Q}$ lying below $\mathfrak{p}$. For $\mathfrak{a} \in I_{K,\mathfrak{m}}$, we define

$$\chi(\mathfrak{a}) = \left(\frac{\varepsilon(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}^{-a} \varepsilon(\mathfrak{a}),$$

where $\left(\frac{\cdot}{\mathfrak{p}}\right)_{L,w_L}$ is the w_L-th power residue symbol in the field L. We remark that if $\mathfrak{a} \in I_{K,\mathfrak{m}}$, then $(\varepsilon(\mathfrak{a}), \mathfrak{p}) = 1$ and thus $\left(\frac{\varepsilon(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}$ makes sense. We show first that χ is independent of the choice of $\varepsilon(\mathfrak{a})$. Indeed, if $\varepsilon'(\mathfrak{a})$ is another element in $L^\times$ satisfying points (1) and (2), then they differ by a root of unity $\zeta(\mathfrak{a}) \in \mu_L$ because of Kronecker's theorem. Therefore, we have

$$\begin{aligned}
\left(\frac{\varepsilon'(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}^{-a} \varepsilon'(\mathfrak{a}) &= \left(\frac{\zeta(\mathfrak{a})\varepsilon(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}^{-a} \zeta(\mathfrak{a})\varepsilon(\mathfrak{a}) \\
&= \zeta(\mathfrak{a})^{-a\frac{\mathbb{N}(\mathfrak{p})-1}{w_L}+1} \left(\frac{\varepsilon(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}^{-a} \varepsilon(\mathfrak{a}) \\
&= \left(\frac{\varepsilon(\mathfrak{a})}{\mathfrak{p}}\right)_{L,w_L}^{-a} \varepsilon(\mathfrak{a}).
\end{aligned}$$

A similar computation shows that $\chi : I_{K,\mathfrak{m}} \longrightarrow L^\times$ is a group morphism. We will now show that there exists $N \geq 1$ such that if $\lambda \in K^\times\left(\mathfrak{m}^N\right)$, then $\chi((\lambda)) = \lambda^\gamma$. This will show that χ is a Hecke character. We showed that the values of χ are independent of the choice of $\varepsilon(\mathfrak{a})$; hence, we have

$$\chi((\lambda)) = \left(\frac{\varepsilon((\lambda))}{\mathfrak{p}}\right)_{L,w_L}^{-a} \varepsilon(\mathfrak{a}) = \left(\frac{\lambda^\gamma}{\mathfrak{p}}\right)_{L,w_L}^{-a} \lambda^\gamma,$$

since $\phi_\gamma((\lambda)) = (\lambda^\gamma)$ and λ^γ satisfies (1) and (2). Letting $\mathfrak{m}_L = p \cdot O_L$, we remark that $N_\gamma(K^\times(\mathfrak{m}^N)) \subseteq L^\times(\mathfrak{m}_L^N)$ for any integer $N \geq 1$. Hence, it suffices to show that there exists $N \geq 1$ such that if $\beta \in L^\times\left(\mathfrak{m}_L^N\right)$ then

$$\left(\frac{\beta}{\mathfrak{p}}\right)_{L,w_L} = 1.$$

But this last equality is true if and only if β is a w_L-th power in $L_\mathfrak{p}^\times$, where $L_\mathfrak{p}$ is the completion of L at $\mathfrak{p}$. The existence of the integer N follows from Lemma 2.24 below.

The existence of an idelic Hecke character having the desired properties follows from §2.5. □

Lemma 2.23. *A number field L has infinitely many prime ideals $\mathfrak{p}$ of absolute degree 1 such that*

(1) $\left(\frac{\mathbb{N}(\mathfrak{p})-1}{w_L}, w_L\right) = 1$,
(2) $\mathbb{N}(\mathfrak{p}) \equiv 1 \pmod{w_L}$.

Proof. See Lemma 4 of [6]. □

Lemma 2.24. *Let L be a number field and $\mathfrak{p}$ a prime ideal of L. We denote a uniformizer at $\mathfrak{p}$ by $\pi_{\mathfrak{p}} \in O_{\mathfrak{p}}$. Let $m \geq 1$ be an integer, then there exists an integer $N \geq 1$ such that $U_{\mathfrak{p}}^{(N)} \subseteq U_{\mathfrak{p}}^{m}$, where $U_{\mathfrak{p}}^{m}$ is the subgroup of $U_{\mathfrak{p}}$ consisting of m-th powers of units and $U_{\mathfrak{p}}^{(N)}$ is the subgroup of $U_{\mathfrak{p}}$ consisting of units u satisfying $u \equiv 1 \mod \pi_{\mathfrak{p}}^{N}$.*

Proof. We refer to the paragraph following Proposition 2 in Chapter 2, §2 of [3] for a proof. □

3. Iwasawa's theorem

We can now formulate our generalized version of Iwasawa's theorem. This theorem is useful, because it says that in particular cases, point (2) of Theorem 2.22 is automatically satisfied and does not have to be checked in order to guarantee the existence of an idelic Hecke character. Hence, in some cases, one just has to check the two conditions

(1) $\gamma \in \mathrm{Im}(\delta) \cap Z_K(L)$,
(2) $\phi_\gamma(I_K) \subseteq P_L$,

in order to get the inclusion $\gamma \in \mathrm{Im}(\delta_L)$.

Let K and L be number fields. Following Iwasawa, we define the group

$$A_{K,L} = \{\gamma \in \mathrm{Im}(\delta) \cap Z_K(L) \,|\, \phi_\gamma(I_K) \subseteq P_L\}.$$

Corollary 2.21 implies that $\mathrm{Im}(\delta_L) \subseteq A_{K,L}$. If K does not contain a CM-field or $L = \mathbb{Q}$, then $\mathrm{Im}(\delta_L) = A_{K,L} = \mathbb{Z} \cdot T_K$. *Hence, throughout §3, we assume that K is a number field containing a CM-field, and we will also assume that L is a CM-field, because of Corollary 2.17.* We also let N be the smallest Galois extension of $\mathbb{Q}$ in $\bar{\mathbb{Q}}$ containing both K and L.

Let E_+ denote the subgroup of E_{L^+} consisting of totally positive units and set $\bar{E}_L = E_+/E_L^{1+j}$, where j is the unique complex conjugation of L. Iwasawa proceeds and defines a pairing $A_{K,L} \times Cl_K \longrightarrow \bar{E}_L$ as follows. Given $\gamma \in A_{K,L}$, it follows from Theorem 2.7 that there exists an integer $\omega \in \mathbb{Z}$ such that $(1 + j')\gamma = \omega T_K$, for all complex conjugations $j' \in \mathrm{Gal}(N/\mathbb{Q})$. Moreover, for all fractional ideals $\mathfrak{a}$ of K, there exists $\alpha \in L^\times$ such that $\phi_\gamma(\mathfrak{a}) = (\alpha)$ by definition of $A_{K,L}$. Hence,

$$(\alpha^{1+j}) = \phi_\gamma(\mathfrak{a})^{(1+j)} = (\mathbb{N}(\mathfrak{a})^\omega).$$

Given $\gamma \in A_{K,L}$ and $\mathfrak{a} \in I_K$, Iwasawa defines $[\gamma, \mathfrak{a}] := \mathbb{N}(\mathfrak{a})^{-\omega}\alpha^{1+j} \in E_+$. Using the group morphism (3), it is simple to check that this correspondence induces a well-defined pairing $A_{K,L} \times I_K \longrightarrow \bar{E}_L$. Moreover, if $\mathfrak{a}$ is a principal ideal, then $[\gamma, \mathfrak{a}] = 1$. Hence, we obtain a pairing

$$A_{K,L} \times Cl_K \longrightarrow \bar{E}_L,$$

defined by $(\gamma, [\mathfrak{a}]) \mapsto [\gamma, \mathfrak{a}]$. Let us recall that the following theorem was proved by Iwasawa under some restrictive hypotheses (see §1).

Theorem 3.1. *The abelian group $\mathrm{Im}(\delta_L)$ is the annihilator of Cl_K in $A_{K,L}$ in the above pairing so that there is a monomorphism*

$$A_{K,L}/\mathrm{Im}(\delta_L) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(Cl_K, \bar{E}_L).$$

Proof. Because of point (1) of Theorem 2.22, it is clear that $\mathrm{Im}(\delta_L)$ is contained in the annihilator of Cl_K. Conversely, suppose that $\gamma \in A_{K,L}$ is such that $[\gamma, C] = 1$, for all $C \in Cl_K$. This means that given a fractional ideal $\mathfrak{a} \in I_K$, there exists a unit $u(\mathfrak{a}) \in E_L$ such that $\mathbb{N}(\mathfrak{a})^{-\omega}\alpha^{1+j} = u(\mathfrak{a})^{1+j}$, where $\phi_\gamma(\mathfrak{a}) = (\alpha)$ as above. Let $\varepsilon(\mathfrak{a}) = u(\mathfrak{a})^{-1}\alpha$. Then $\phi_\gamma(\mathfrak{a}) = (\varepsilon(\mathfrak{a}))$ and $|\varepsilon(\mathfrak{a})|_v = \mathbb{N}(\mathfrak{a})^\omega$ for all infinite places v of L. It follows from Theorem 2.22 that there exists $\mu \in H_K(L)$ whose infinity-type is γ. This means that $\gamma \in \mathrm{Im}(\delta_L)$ as we wanted to show. □

Corollary 3.2. *If h_K is odd or $E_+ = E_L^{1+j}$, then $A_{K,L} = \mathrm{Im}(\delta_L)$.*

Proof. This is clear, since $\bar{E}_L$ is annihilated by 2. □

References

[1] P. Deligne. *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[2] Kenkichi Iwasawa. Some remarks on Hecke characters. In *Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976)*, pages 99–108. Japan Soc. Promotion Sci., Tokyo, 1977.

[3] Serge Lang. *Algebraic number theory*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont., 1970.

[4] Ryotaro Okazaki. Inclusion of CM-fields and divisibility of relative class numbers. *Acta Arith.*, 92(4):319–338, 2000.

[5] Claus-Günther Schmidt. *Arithmetik abelscher Varietäten mit komplexer Multiplikation*, volume 1082 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984. With an English summary.

[6] Goro Shimura. On the zeta-function of an abelian variety with complex multiplication. *Ann. of Math. (2)*, 94:504–533, 1971.

[7] André Weil. On a certain type of characters of the idèle-class group of an algebraic number-field. In *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955*, pages 1–7, Tokyo, 1956. Science Council of Japan.

[8] Tong-Hai Yang. Existence of algebraic Hecke characters. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(12):1041–1046, 2001.