UNIVERSITY OF CALIFORNIA, SAN DIEGO

On a generalization of the rank one Rubin-Stark conjecture

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by

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The dissertation of Daniel Vallières is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2011

À mes parents: Michelle et Lucien.

A une époque où les approches du temple du savoir
ressemblent quelquefois à celles de la bourse,
où l'on y court le risque d'être étourdi
par des troupes bruyantes de marchands et de bateleurs,
nous avions besoin de nous affermir dans notre croyance
qu'une vie donnée toute à la science est encore possible,
sans faiblesses ni compromissions ...
—André Weil

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Speaking of Max Dehn, André Weil wrote in his autobiography: *"A humanistic mathematician, who saw mathematics as one chapter – certainly not the least important – in the history of the human mind..."* I like to think the same way. It is thus a pleasure for me to take some time in order to thank several people, who have helped me becoming what I am as a human being, and whose influence can be seen in my way of doing mathematics.

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ABSTRACT OF THE DISSERTATION

On a generalization of the rank one Rubin-Stark conjecture

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This thesis is concerned with the *extended abelian rank one Stark conjecture* stated for the first time in Erickson's thesis, see [19]. Here, we investigate it in more depth then has been done so far. We formulate a *stronger question* (Question 5.2) which seems easier to investigate both theoretically and computationally. Question 5.2 includes a *generalization of the Brumer-Stark conjecture* on annihilation of class groups (see Question 5.7). We link it with a conjecture of Gross and in the process find some *new integrality properties* of the Stickelberger element (Theorem 5.29). In order to better understand the notion of 1-cover already introduced in [19], we single out the notion of *minimal cocyclic subgroups*, which is purely group theoretical (Definition 4.17). Finally, using these minimal cocyclic subgroups, we provide some *numerical examples* with base field $\mathbb{Q}$ for which Question 5.2, and thus the extended abelian rank one Stark conjecture, have an affirmative answer (Appendix B).

Chapter 1

Introduction

In a series of papers in the 70s and early 80s, Harold Stark formulated a question which is now referred to as *Stark's conjecture.* In his fourth paper on the subject ([53]), he stated a more precise conjecture for abelian L-functions with order of vanishing one at $s = 0$ which is now known as the *abelian rank one Stark conjecture.* It belongs to number theory, or more precisely, to the field of *special values of L-functions*, and can be viewed as a special case of the very general *Equivariant Tamagawa Number Conjecture*, but is rather appealing because of its very concrete formulation. Despite some progress made towards its solution in the last decade, it is still wide open and calls for further investigation and new ideas.

The abelian rank one Stark conjecture is stated under some hypotheses: The set S of primes involved has to contain the infinite ones, the ramified ones, *at least one prime which splits completely*, and be of cardinality greater than or equal to two. These last two conditions *guarantee that all the S-imprimitive L-functions vanish at $s = 0$ with a zero of order at least one.* On the other hand, there are some abelian extensions of number fields and some sets S of primes for which all the S-imprimitive L-functions vanish at $s = 0$ with order of vanishing at least one, but such that S does not contain any prime which splits completely. Stark formulated a question in this setting as well, which in the case of a positive answer also predicts the existence of a unit, but the χ-regulator involved is more complicated. Very little evidence has been found so far to support a positive answer. It is not even known for all abelian extensions of $\mathbb{Q}$.

In Erickson's thesis, summarized in [19], one can find a formulation of the *extended abelian rank one Stark conjecture* and a positive answer in some cases. Emmons and Popescu (see [17]) extended the conjecture to any order of vanishing very much in the spirit of the Rubin-Stark conjecture described in [42].

In this thesis, we pursue the study of the extended abelian rank one Stark conjecture with the hope of providing further evidence for it. Chapter 2 and 3 introduce the classical abelian rank one Stark conjecture. We spent quite some time trying to prove the conjecture over $\mathbb{Q}$ using *cyclotomic units and Gauss sums* without any success. On the other hand, there are really few cases where one can actually play with Stark units explicitly and when the base field is $\mathbb{Q}$, such an explicit description is possible. Thus, we explain the relevant concepts of Gauss sums and cyclotomic units in Chapter 2. In Chapter 4, we recall the different formulations of the extended abelian rank one Stark conjecture contained in Erickson and Emmons's theses. Chapter 5 could be considered as the core of this dissertation. We investigate the conjecture and in the process we formulate a *stronger question* which is satisfied in all cases which have been studied both theoretically and numerically. The reader will also find some *numerical examples* with base field $\mathbb{Q}$ in Appendix B.

1.1 Notations

Some notations will be introduced along the way, but we record here those which will be used without any warning in the text.

§I The letter K will usually stand for a number field, and O_K will be its ring of integers. We shall use the Fraktur typeface $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ for fractional ideals of K and reserve $\mathfrak{p}, \mathfrak{q}, \mathfrak{P}$ for prime ideals of K. An equivalence class of absolute values of K will be called a place or a prime even though this latter denomination is rather misleading since there is no prime ideal associated to the archimedean places. We will use the usual abuse of notation for the places (archimedean or not) of K and denote them by letters like v and w. The pairs of words finite or non-archimedean and infinite or archimedean will be used synonymously. Principal ideals will be denoted by $\alpha \cdot O_K$ or (α) if the field K is understood from the context. If A is a subgroup of $K^\times$, we denote by (A) the group of principal ideals generated by elements of A. The group of fractional ideals is denoted by I_K and the subgroup consisting of the principal ones by $P_K = (K^\times)$. It is a fundamental result of algebraic number theory that the class group $Cl_K = I_K/P_K$ is finite; its cardinality is denoted by h_K. We use E_K to denote the units of K; its rank over $\mathbb{Z}$ was determined by Dirichlet and is given by $r = r_1 + r_2 - 1$ where r_1 is the number of real embeddings of K and r_2 is half the number of complex embeddings of K. We have $[K:\mathbb{Q}] = r_1 + 2r_2$. The symbol R_K stands for the regulator of K. The finite group of roots of unity in K is denoted by μ_K and its cardinality by w_K. More generally, if S is a finite set of primes containing the infinite ones, the ring of S-integers is denoted by $O_{K,S}$ and the group of S-units by $E_{K,S}$. If S is any set of places of K, we use the notation S_0 (resp. S_∞) for the finite ones (resp. infinite ones) in S, thus $S = S_0 \cup S_\infty$. The ring $O_{K,S}$ is a Dedekind domain, since it is the ring of fractions $T^{-1}O_K$ where T consists of the union of all finite primes not in S. Hence, we can talk about its class group $Cl(O_{K,S})$ and we have an exact sequence

$$1 \longrightarrow E_K \longrightarrow E_{K,S} \longrightarrow \bigoplus_{\mathfrak{p}\in S_0} K^\times/O_{K,\mathfrak{p}}^\times \longrightarrow Cl_K \longrightarrow Cl(O_{K,S}) \longrightarrow 1,$$

which shows that the rank of $E_{K,S}$ is $|S|-1$ since $K^\times/O_{K,\mathfrak{p}}^\times \simeq \mathbb{Z}$. The S-regulator is denoted by $R_{K,S}$. Moreover $Cl(O_{K,S})$ is finite, and its cardinality is denoted by $h_{K,S}$. Another invariant associated to a number field K is its discriminant $\Delta_K \in \mathbb{Z}$; a prime p of $\mathbb{Q}$ ramifies in K if and only if it divides Δ_K. Given a fractional ideal $\mathfrak{a}$ of K, we denote its absolute norm by $\mathbb{N}(\mathfrak{a})$. If $\mathfrak{p}$ is a prime ideal of K, the residue field $O_K/\mathfrak{p}$ is denoted by $\overline{K}_\mathfrak{p}$; its cardinality is given by $\mathbb{N}(\mathfrak{p})$. If we need to, we will denote the algebraic closure of a field K by K^{alg} rather than the standard notation $\overline{K}$ in order to avoid confusion with the residue field.

§II If K/k is a finite extension of number fields, we denote the norm map both at the level of field elements and ideals by $N_{K/k}$. For a principal ideal $\alpha \cdot O_K$, we have the following equality of ideals in k:

$$N_{K/k}((\alpha)) = (N_{K/k}(\alpha)). \tag{1.1}$$

Moreover, if the base field is $\mathbb{Q}$, the absolute norm of an ideal $\mathbb{N}(\mathfrak{a})$ is nothing else than the positive generator of $N_{K/\mathbb{Q}}(\mathfrak{a})$. The set of places of k which are either archimedean or ramified in K/k will be denoted by $S(K/k)$; by a classical theorem in algebraic number theory, it is a finite set of places of k. If S is a set of places of k, we refer to the set of places of K lying above the ones in S by S_K. For simplicity, we agree on the following convention: Rather than E_{K,S_K}, O_{K,S_K}, and so on, we will write $E_{K,S}, O_{K,S}$, etc.

§III If v is a finite place corresponding to the prime ideal $\mathfrak{p}$, we denote the normalized valuation corresponding to it by $\mathrm{ord}_{\mathfrak{p}}$. The symbol $|\cdot|_v$ will stand for the normalized absolute value corresponding to the place v. It is given by the following formulae:

$$|x|_v = \begin{cases} \mathbb{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}, & \text{if } v \text{ is a finite place corresponding to the prime ideal } \mathfrak{p}, \\ |\tau(x)|, & \text{if } v \text{ is a real place corresponding to the real embedding } \tau, \\ |\tau(x)|^2, & \text{if } v \text{ is a complex place corresponding to a complex embedding } \tau. \end{cases}$$

We have the product formula

$$\prod_v |x|_v = 1,$$

for all $x \in K^\times$ which follows from (1.1) when the base field k is $\mathbb{Q}$. The completion of a number field K at a place v is denoted by K_v.

§IV If K/k is a Galois extension of number fields with Galois group G, and w is a place of K, we denote the decomposition group by G_w and the inertia group by I_w. If w lies above v, then we have the usual short exact sequence

$$1 \longrightarrow I_w \longrightarrow G_w \longrightarrow \mathrm{Gal}(\overline{K}_w/\overline{k}_v) \longrightarrow 1.$$

If w is unramified, its Frobenius automorphism is denoted by σ_w. If $w = \mathfrak{P}$ is a finite prime lying above $v = \mathfrak{p}$, then the Frobenius automorphism is characterized by the property

$$x^{\sigma_{\mathfrak{P}}} \equiv x^{\mathbb{N}(\mathfrak{p})} \pmod{\mathfrak{P}},$$

for all $x \in O_K$. The Frobenius automorphism is also denoted by $(\mathfrak{P}, K/k)$ or

$$\left(\frac{K/k}{\mathfrak{P}}\right).$$

It is well-known that when the extension is abelian, the Frobenius automorphism does not depend on the choice of $\mathfrak{P}$ lying above $\mathfrak{p}$. In this case, it is denoted by $\sigma_{\mathfrak{p}}$ rather than $\sigma_{\mathfrak{P}}$.

§V Two particular kinds of abelian extensions of number fields are particularly important: The cyclotomic and the Kummer extensions. Given a number field K, a cyclotomic extension is an extension of K obtained by adjoining roots of unity, i.e. a field of the form $K(\zeta)$ where ζ is some primitive root of unity in some algebraic closure of K. When the base field is $\mathbb{Q}$, the fields $\mathbb{Q}(\zeta)$ are referred to as cyclotomic fields. The extensions $K(\zeta)/K$ are Galois with Galois group isomorphic to a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ if ζ is a primitive m-th root of unity. While trying to generalize the quadratic reciprocity law, Kummer was led to study some particular cyclic extensions of cyclotomic fields. These extensions and the ones obtained from them by taking compositum of fields are now called Kummer extensions; they are the abelian extensions of K with exponent a divisor of w_K. Since these extensions show up frequently in connection with the abelian rank one Stark conjecture, we included parts of the theory which are used in this thesis in Appendix A.

§VI We denote the complex conjugation of $\mathbb{C}$ by ρ. It is an element of order 2 in $\mathrm{Aut}(\mathbb{C})$. A CM-field is a totally imaginary number field which is a quadratic extension of a totally real field. If K is a CM-field we denote its maximal real subfield by K^+. CM stands for complex multiplication and those fields appear in the theory of complex multiplication of abelian varieties due to Shimura, Taniyama, and Weil. Examples of CM-fields are provided by the quadratic imaginary number number fields and the cyclotomic number fields, but these do not exhaust all of them. If K is a CM-field, the complex conjugation $\rho \in \mathrm{Aut}(\mathbb{C})$ induces a well-defined automorphism $j \in \mathrm{Gal}(K/K^+)$. It is in fact in the center of $\mathrm{Aut}_{\mathbb{Q}}(K)$ and it satisfies

$$\tau \cdot j = \rho \cdot \tau,$$

for all embeddings τ of K in $\mathbb{C}$.

§VII Because of the distinguished role played by cyclotomic fields we adopt a particular notation for them. Let ζ_m be a primitive m-th root of unity in some algebraic closure of $\mathbb{Q}$. The field $\mathbb{Q}(\zeta_m)$ is an abelian Galois extension over $\mathbb{Q}$. Its Galois group over $\mathbb{Q}$ is denoted by G_m; we have $G_m \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. If t is an integer satisfying $(t, m) = 1$, we refer to the automorphism $\zeta_m \mapsto \zeta_m^t$ of G_m by σ_t. If we want to see $\mathbb{Q}(\zeta_m)$ inside of $\mathbb{C}$, we will usually do it through the embedding $\zeta_m \mapsto \exp(2\pi i/m)$ unless otherwise specified. We also remind the reader that $\mathbb{Q}(\zeta_m)$ has $2m$ roots of unity if m is odd, and m roots of unity if m is even.

§VIII Class field theory will be used throughout this thesis. Let k be a number field and let T be a finite set of finite primes, we denote by $k_T^\times$ the set of elements in $k^\times$ which are relatively prime with T. It is a subgroup of $k^\times$. If $\mathfrak{m}_\infty$ is a set of real embeddings of k, then $k^\times_{T,\mathfrak{m}_\infty}$ will be the subgroup of $k^\times_T$ consisting of elements which are positive at the real embeddings in $\mathfrak{m}_\infty$. A modulus $\mathfrak{m} = \mathfrak{m}_0 \cdot \mathfrak{m}_\infty$ of k, consists of a formal product of an integral ideal $\mathfrak{m}_0$ and a set of real embeddings $\mathfrak{m}_\infty$ of k. The set of primes dividing $\mathfrak{m}_0$ will be denoted by $\mathrm{supp}(\mathfrak{m}_0)$. The group of fractional ideals of k relatively prime with $\mathfrak{m}_0$ is denoted by $I_{k,\mathfrak{m}}$ (the infinite part $\mathfrak{m}_\infty$ plays no role in this definition). The symbol $k^\times_\mathfrak{m}$ denotes all elements of $k^\times$ which satisfy $\alpha \equiv 1 \mod {}^\times\mathfrak{m}$. These last congruences are the multiplicative congruences of Hasse. For the convenience of the reader, we remind here what they mean. Saying that $\alpha \equiv 1 \mod {}^\times\mathfrak{m}$ amounts to saying that

- $\mathrm{ord}_\mathfrak{p}(\alpha - 1) \geq \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{m}_0)}$, for all $\mathfrak{p} \mid \mathfrak{m}_0$ and
- $\tau(\alpha) \geq 0$ for all $\tau \in \mathfrak{m}_\infty$.

The set $k^\times_\mathfrak{m}$ is a subgroup of $k^\times_{T,\mathfrak{m}_\infty}$, where $T = \mathrm{supp}(\mathfrak{m}_0)$. The set of principal ideals (α) where $\alpha \in k^\times_\mathfrak{m}$ is denoted by $P_{k,\mathfrak{m}}$; it is a subgroup of $I_{k,\mathfrak{m}}$. The quotient $Cl_{k,\mathfrak{m}} = I_{k,\mathfrak{m}}/P_{k,\mathfrak{m}}$ is the ray class group modulo $\mathfrak{m}$. The units inside of $k^\times_\mathfrak{m}$ form a subgroup in E_k which is denoted by $E_{k,\mathfrak{m}}$ and its torsion subgroup is denoted by $\mu_{K,\mathfrak{m}}$. In other words, $E_{k,\mathfrak{m}} = E_k \cap k^\times_\mathfrak{m}$. If $\mathfrak{m} = O_k$, we recover the usual class group and group of units. If $T = \mathrm{supp}(\mathfrak{m}_0)$, we have two short exact sequences

$$1 \longrightarrow (k^\times_T)/(k^\times_\mathfrak{m}) \longrightarrow Cl_{k,\mathfrak{m}} \longrightarrow Cl_K \longrightarrow 1,$$

and

$$1 \longrightarrow E_k/E_{k,\mathfrak{m}} \longrightarrow k^\times_T/k^\times_\mathfrak{m} \longrightarrow (k^\times_T)/(k^\times_\mathfrak{m}) \longrightarrow 1,$$

which combined together gives the following exact sequence

$$1 \longrightarrow E_{k,\mathfrak{m}} \longrightarrow E_k \longrightarrow k^\times_T/k^\times_\mathfrak{m} \longrightarrow Cl_{k,\mathfrak{m}} \longrightarrow Cl_k \longrightarrow 1.$$

Using the isomorphism

$$k^\times_T/k^\times_\mathfrak{m} \simeq (O_k/\mathfrak{m}_0)^\times \times \prod_{\tau \in \mathfrak{m}_\infty} \{\pm 1\},$$

we conclude that both $[E_k : E_{k,\mathfrak{m}}]$ and $Cl_{k,\mathfrak{m}}$ are finite. The cardinality of the ray class group $Cl_{k,\mathfrak{m}}$ is denoted by $h_{k,\mathfrak{m}}$. The exact sequence above shows more precisely that

$$h_{k,\mathfrak{m}} = h_k \cdot \frac{\varphi(\mathfrak{m}_0)}{[E_k : E_{k,\mathfrak{m}}]} \cdot 2^{|\mathfrak{m}_\infty|},$$

where $\varphi(\mathfrak{m}_0) = |(O_k/\mathfrak{m}_0)^\times|$ is the generalization of the usual Euler φ-function.

Given any modulus $\mathfrak{m}$, class field theory shows the existence of a finite abelian extension of k denoted by $k_\mathfrak{m}$, called the ray class field of k modulo $\mathfrak{m}$, which is unramified outside of $\mathrm{supp}(\mathfrak{m}_0)$ and $\mathfrak{m}_\infty$ and such that the Artin map

$$I_{k,\mathfrak{m}} \longrightarrow \mathrm{Gal}(k_\mathfrak{m}/k),$$

induces an isomorphism

$$Cl_{k,\mathfrak{m}} \simeq \mathrm{Gal}(k_\mathfrak{m}/k).$$

Be aware that $k_\mathfrak{m}$ denotes a ray class field of k and $k_\mathfrak{m}^\times$ denotes a subgroup of $k^\times$. It should be clear from the context which one we are alluding too.

Class field theory also shows that every abelian extension of k is contained in a ray class field for some modulus $\mathfrak{m}$. More precisely if K/k is a finite abelian extension, then there exists a modulus $\mathfrak{m}$, not necessarily unique, such that all the ramified primes in K/k divide $\mathfrak{m}$, and such that the Artin map

$$I_{k,\mathfrak{m}} \longrightarrow \mathrm{Gal}(K/k),$$

is surjective. Moreover, the kernel of the Artin map is given by the Takagi group

$$T_\mathfrak{m}(K/k) = N_{K/k}(I_{K,\mathfrak{m}}) \cdot P_{k,\mathfrak{m}}.$$

In this case, K is contained in $k_\mathfrak{m}$. Given an abelian extension K/k the smallest modulus $\mathfrak{m}$ (with respect to divisibility) for which $K \subseteq k_\mathfrak{m}$ is called the conductor of K/k and is denoted by $\mathfrak{f}(K/k)$. A prime (finite or infinite) divides $\mathfrak{f} = \mathfrak{f}(K/k)$ if and only if it ramifies in K/k. Moreover, the Artin map induces an isomorphism

$$I_{k,\mathfrak{f}}/T_\mathfrak{f}(K/k) \xrightarrow{\simeq} \mathrm{Gal}(K/k).$$

The conductor of the ray class field modulo $\mathfrak{m}$ might be smaller than $\mathfrak{m}$. The Artin map is denoted by $(\mathfrak{a}, K/k)$ where $\mathfrak{a}$ is any fractional ideal for which this symbol makes sense.

In order to get a one-to-one correspondence between finite abelian extensions of k and some objects associated to k, one has to introduce ideal groups modulo $\mathfrak{m}$ which are subgroups H of $I_{k,\mathfrak{m}}$ satisfying

$$P_{k,\mathfrak{m}} \subseteq H \subseteq I_{k,\mathfrak{m}},$$

and introduce an equivalence relation on them since an abelian extension can be contained in more than one ray class fields. These details will not matter much to us so we do not explain them.

The ray class field modulo $\mathfrak{m} = O_k$ is called the Hilbert class field of k and its Galois group is isomorphic to Cl_k. It is usually denoted by H_k rather than k_{O_k}. It can be shown that it is the maximal unramified abelian extension of k (unramified everywhere including the infinite places). It is sometimes referred to as the small Hilbert class field or the ordinary Hilbert class field to distinguish it from the ray class field modulo $\mathfrak{m} = O_k \cdot \mathfrak{m}_\infty$, denoted by H_k^+, where $\mathfrak{m}_\infty$ is the set of all real embeddings of k. This latter ray class field is also unramified at finite primes, but ramification at infinite places is allowed; it is called the big or the narrow Hilbert class field. The word narrow might lead to confusion since in fact $H_k \subseteq H_k^+$ and we will avoid this terminology. If k is totally complex, there is no distinction between H_k and H_k^+.

§IX If S a finite set of finite primes and $\mathfrak{m}$ is a modulus such that $\mathrm{supp}(\mathfrak{m}_0) \cap S = \varnothing$, then we will denote the maximal S-split extension (i.e. the primes in S split completely) inside of $k_\mathfrak{m}$ by $k_\mathfrak{m}^S$. The Artin map induces an isomorphism

$$I_{k,\mathfrak{m}}/P_{k,\mathfrak{m}} \cdot < S > \xrightarrow{\simeq} \mathrm{Gal}(k_\mathfrak{m}^S/k),$$

where $< S >$ denotes the group of fractional ideals generated by primes in S. The class group $I_{k,\mathfrak{m}}/P_{k,\mathfrak{m}} \cdot < S >$ will be denoted by $Cl_{k,\mathfrak{m}}^S$ and referred to as the $(S, \mathfrak{m})$-class group. If we introduce the $(S, \mathfrak{m})$-units

$$E_{k,\mathfrak{m}}^S = \{x \in k_\mathfrak{m}^\times \,|\, \mathrm{ord}_\mathfrak{p}(x) = 0, \text{ for all finite } \mathfrak{p} \notin S\},$$

we then have the following exact sequence

$$1 \longrightarrow E^S_{k,\mathfrak{m}} \longrightarrow E^S_k \longrightarrow (O_k/\mathfrak{m}_0)^\times \times \prod_{\tau\in\mathfrak{m}_\infty} \{\pm 1\} \longrightarrow Cl^S_{k,\mathfrak{m}} \longrightarrow Cl^S_k \longrightarrow 1.$$

Remark that if S is a finite set of places containing the infinite ones then $E^{S_0}_k \simeq E_{k,S}$ and $Cl^{S_0}_k \simeq Cl(O_{k,S})$. To each group of units $E^S_{k,\mathfrak{m}}$ is associated a regulator $R_{k,S,\mathfrak{m}}$ in the usual way.

§X Two cases of §IX will show up frequently in this thesis. If S is a finite set of places of k containing the infinite places, the S-class group $Cl(O_{k,S}) \simeq Cl^{S_0}_k$ will simply be denoted by $Cl_{k,S}$ or Cl_S if the field k is clear from the context. If we are also given another finite set of finite primes T satisfying $S \cap T = \varnothing$, then we set

$$\mathfrak{m}(T) = \prod_{\mathfrak{p}\in T} \mathfrak{p},$$

and refer to $\mathfrak{m}(T)$ as the tame modulus associated to k. The ray class field $k_{\mathfrak{m}(T)}$ is the maximal T-tamely ramified extension of k unramified away from T and the extension $k^{S_0}_{\mathfrak{m}(T)}$ is the maximal S_0-split and T-tamely ramified extension of k unramified away from T. The class group $Cl^{S_0}_{k,\mathfrak{m}(T)}$ will be referred to as the (S,T)-class group and will be denoted more briefly by $Cl_{k,S,T}$. We will also refer to the (S,T)-units as $E_{k,S,T}$ and its torsion group as $\mu_{k,T}$, since it does not depend on S. We denote $|\mu_{k,T}|$ by $w_{k,T}$. The (S,T)-regulator is denoted similarly by $R_{k,S,T}$.

§XI It might be useful to specialize class field theory to the case where the base field is $\mathbb{Q}$. In this case, the moduli are simply denoted by $m \cdot \infty$ rather than $(m) \cdot \infty$. The ray class field modulo $m \cdot \infty$ is $\mathbb{Q}(\zeta_m)$ and the ray class field modulo m is $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$, the maximal real subfield of $\mathbb{Q}(\zeta_m)$. If m is odd, then $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{2m})$ which gives an example of a ray class field modulo $\mathfrak{m}$ for which its conductor properly divides $\mathfrak{m}$. If m is odd or $m \not\equiv 2 \pmod 4$, then the conductor of $\mathbb{Q}(\zeta_m)$ is precisely $m \cdot \infty$. The fact that every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_m)$ (a ray class field) is known as the Kronecker-Weber theorem. We have

$$Cl_{\mathbb{Q},m\cdot\infty} \simeq (\mathbb{Z}/m\mathbb{Z})^\times,$$

and

$$Cl_{\mathbb{Q},m} \simeq (\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}.$$

One fundamental difference between the general case and the case where the base field is $\mathbb{Q}$ is that we know generators for the ray class fields, namely the roots of unity. The action of the Artin symbol on the generators is also known explicitly: If p does not divide m then it is unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ and the Frobenius automorphism $(p, \mathbb{Q}(\zeta_m)/\mathbb{Q})$ is given by

$$\zeta_m \mapsto \zeta_m^p.$$

§XII If G is a finite abelian group and M a $\mathbb{Z}[G]$-module, we will write $\mathbb{C}M$ rather than $\mathbb{C}[G] \otimes_{\mathbb{Z}[G]} M$. As usual, M^G will denote the fixed points of M under the action of G.

Chapter 2

Classical results

2.1 Stickelberger's Theorem

The goal of this section is to remind the reader about Stickelberger's Theorem. It might be appropriate to start with the celebrated quadratic reciprocity law

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}},$$

where p and q are odd primes and $(\frac{\cdot}{\cdot})$ is the Legendre symbol, which we recall is defined as

$$\left(\frac{n}{p}\right) = \begin{cases} 0, \text{ if } p|n, \\ 1, \text{ if } n \text{ is a square modulo } p, \\ -1, \text{ otherwise.} \end{cases}$$

The only thing we need to know about this symbol is the Euler identity

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p},$$

whenever p is an odd prime. This identity can be proved as follows. If $p \,|\, a$ then it is clear, otherwise if there exists $x \in \mathbb{Z}$ such that $x^2 \equiv a \pmod{p}$, we have

$$a^{\frac{p-1}{2}} \equiv x^{p-1} \equiv 1 \pmod{p}.$$

Conversely, if $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$, then write $a = \gamma^s$ for some integers γ, s where the class of γ modulo p generates $(\mathbb{Z}/p\mathbb{Z})^\times$. Then

$$a^{\frac{p-1}{2}} \equiv \gamma^{\frac{s(p-1)}{2}} \equiv 1 \pmod{p},$$

from which we conclude that there exists $t \in \mathbb{Z}$ such that

$$\frac{s(p-1)}{2} = t(p-1).$$

Letting $x = \gamma^{\frac{s}{2}}$, we get $x^2 \equiv a \pmod{p}$.

As it is well-known to arithmeticians, several proofs of the quadratic reciprocity law have been given. Let us remind one which is due to Gauss and has the advantage of being generalizable in part to some other reciprocity laws. Instrumental in this proof is the so-called Gauss sum

$$g = \sum_{a \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \zeta_p^a,$$

where $\zeta_p = \exp(2\pi i/p)$ and p is an odd prime. Gauss showed that

$$g^2 = (-1)^{\frac{p-1}{2}} p.$$

It is customary to denote this latter quantity by p^*; the Euler identity allows us to write $p^* = \left(\frac{-1}{p}\right) p$. This last equation shows further that the unique quadratic number field inside of $\mathbb{Q}(\zeta_p)$ is $\mathbb{Q}(\sqrt{p^*})$, a non-trivial fact (the uniqueness follows from Galois theory).

If q is another odd prime, we have

$$g^{q-1} = (p^*)^{\frac{q-1}{2}} \equiv \left(\frac{p^*}{q}\right) \pmod{q}, \tag{2.1}$$

by Euler's identity. On the other hand, in the ring $\mathbb{Z}[\zeta_p]$, we have

$$\begin{aligned} g^q = \left(\sum_{a \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \zeta_p^a\right)^q &\equiv \sum_{a \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \zeta_p^{qa} \pmod{q} \\ &\equiv \left(\frac{q}{p}\right)^{-1} g \pmod{q}. \end{aligned}$$

Combining this last equation with (2.1), and using the fact that g is relatively prime to

q in $\mathbb{Z}[\zeta_p]$, we get the identity

$$\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right),$$

which is easily seen to be equivalent to the quadratic reciprocity law. To complete the story, we should also prove the supplementary law (i.e. the computation of $\left(\frac{2}{p}\right)$ where p is an odd prime), but we will not pursue this matter here.

A source of discoveries in number theory has always been to generalize and better understand this quadratic reciprocity law. We shall now generalize the concepts of Gauss sums and Legendre symbol, and see how it leads naturally to Stickelberger's Theorem.

2.1.1 Gauss sums (Lagrange Resolvents)

Let F be a finite field of cardinality $q = p^f$ where p is a prime number. Let χ be a multiplicative character of F, that is a group homomorphism $\chi : F^\times \to \mathbb{C}^\times$, and let ψ be an additive character of F, i.e. a group homomorphism $\psi : F \longrightarrow \mathbb{C}^\times$. A sum of the form

$$G(\chi, \psi) = \sum_{\alpha \in F^\times} \chi(\alpha)\psi(\alpha)$$

is commonly called a *Gauss sum*.

Lemma 2.1. *Let F be a finite field of characteristic p and let $\mathrm{Tr} : F \longrightarrow \mathbb{F}_p$ be the usual trace function where $\mathbb{F}_p$ stands for the finite field of p elements. Then the map $F \longrightarrow \mathbb{C}^\times$ defined by*

$$\alpha \mapsto \zeta_p^{\mathrm{Tr}(\alpha)},$$

is a non-trivial additive character of F. Moreover, any other additive character of F is given by

$$\alpha \mapsto \zeta_p^{\mathrm{Tr}(a \cdot \alpha)},$$

for some $a \in F$.

Proof:

The first part of the lemma follows from the fact that the trace map is surjective. The second part follows simply by checking that the map

$$F \longrightarrow \mathrm{Hom}_{\mathbb{F}_p}(F, \mu_p),$$

defined by $a \mapsto (\alpha \mapsto \zeta_p^{\mathrm{Tr}(a\cdot\alpha)})$ is injective and hence a bijection.

Q.E.D.

Definition 2.2. Given any multiplicative character χ of the finite field F, we let

$$G(\chi) = G_1(\chi) = \sum_{\alpha \in F^\times} \chi(\alpha)\zeta_p^{\mathrm{Tr}(\alpha)},$$

and given any $a \in F$, we define

$$G_a(\chi) = \sum_{\alpha \in F^\times} \chi(\alpha)\zeta_p^{\mathrm{Tr}(a\cdot\alpha)}.$$

Lemma 2.3. *We have*

(1) $G_0(\chi_1) = q - 1$,

(2) *If* $\chi \neq \chi_1$, *then* $G_0(\chi) = 0$,

(3) *If* $a \in F^\times$, *then* $G_a(\chi) = \overline{\chi(a)}G(\chi)$.

Proof:

The first two formulae follow from the orthogonality relations for abelian characters, whereas the last one is a direct computation which is left to the reader.

Q.E.D.

Because of Lemmata 2.1 and 2.3, the study of the sums $G(\chi, \psi)$ for any additive character ψ and a fixed multiplicative character χ, can be reduced to the sole study of $G(\chi)$.

Let m be the order of χ. We remark that $m \mid (q-1)$ and thus $(p, m) = 1$. The Gauss sums clearly lie in $\mathbb{Q}(\zeta_m, \zeta_p) = \mathbb{Q}(\zeta_{pm})$. The next theorem describes the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q})$ on these Gauss sums.

Theorem 2.4. *Let* t *be an integer satisfying* $(t, pm) = 1$.

(1) *Suppose first that* $t \equiv 1 \pmod{m}$, *then*

$$G(\chi)^{\sigma_t} = \overline{\chi(t)} \cdot G(\chi).$$

(2) *On the other hand, if* $t \equiv 1 \pmod p$*, then*

$$G(\chi)^{\sigma_t} = G(\chi^t).$$

Proof:

Let t be an integer satisfying $(t, pm) = 1$ and $t \equiv 1 \pmod m$. Then

$$\begin{aligned} G(\chi)^{\sigma_t} &= \sum_{\alpha \in F^\times} \chi(\alpha)\zeta_p^{t\mathrm{Tr}(\alpha)} \\ &= \sum_{\alpha \in F^\times} \chi(\alpha)\zeta_p^{\mathrm{Tr}(t\alpha)} \\ &= \sum_{\alpha \in F^\times} \overline{\chi(t)}\chi(\alpha)\zeta_p^{\mathrm{Tr}(\alpha)} \\ &= \overline{\chi(t)} \cdot G(\chi). \end{aligned}$$

The proof of the second part is similar and left to the reader.

Q.E.D.

Remark 1. *Since* $\mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q}(\zeta_m)) \times \mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q}(\zeta_p))$*, this last theorem allows one to derive the Galois action of the whole group* $\mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q})$ *on Gauss sums.*

Theorem 2.5. *Given any non-trivial multiplicative character* χ *of the finite field* F*, we have*

$$|G(\chi)|^2 = q.$$

Proof:

The idea is to compute the sum

$$S = \sum_{a \in F} G_a(\chi)\overline{G_a(\chi)},$$

in two different ways. On one hand,

$$S = \sum_{a \in F^\times} \chi(a)\overline{\chi(a)}G(\chi)\overline{G(\chi)} = (q-1)|G(\chi)|^2,$$

by Lemma 2.3, whereas on the second hand

$$S = \sum_{a\in F}\sum_{x\in F^\times}\sum_{y\in F^\times}\chi(x)\overline{\chi(y)}\zeta_p^{\mathrm{Tr}(a(x-y))} = \sum_{x\in F^\times}\sum_{y\in F^\times}\chi(x)\overline{\chi(y)}\sum_{a\in F}\zeta_p^{\mathrm{Tr}(a(x-y))}.$$

Once more, using the orthogonality relations for abelian characters, one gets

$$S = q(q-1).$$

We can then conclude that the desired equality holds true.

Q.E.D.

We finish this subsection with another basic theorem on Gauss sums.

Theorem 2.6. *The complex conjugate of a Gauss sum satisfies*

$$\overline{G(\chi)} = \chi(-1)G(\overline{\chi}).$$

Proof:

We have

$$\begin{aligned}\overline{G(\chi)} &= \overline{\sum_{\alpha\in F^\times}\chi(\alpha)\zeta_p^{\mathrm{Tr}(\alpha)}}\\ &= \sum_{\alpha\in F^\times}\overline{\chi(\alpha)}\zeta_p^{\mathrm{Tr}(-\alpha)}\\ &= \sum_{\alpha\in F^\times}\overline{\chi(-\alpha)}\zeta_p^{\mathrm{Tr}(\alpha)}\\ &= \overline{\chi(-1)}G(\overline{\chi})\\ &= \chi(-1)G(\overline{\chi}),\end{aligned}$$

since $\chi(-1) = \pm 1$.

Q.E.D.

2.1.2 The m-th power residue symbol and Hasse's principle for powers

In this subsection, we explain the m-th power residue symbol, a generalization of the Legendre symbol. Let K be a number field and m any positive integer dividing w_K. Consider a prime $\mathfrak{p}$ of K satisfying $(\mathfrak{p}, m) = 1$. Given any $\alpha \in K^\times$ such that $(\alpha, \mathfrak{p}) = 1$, the m-th power residue symbol $\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}$ is defined as follows. The Kummer extension $K(\sqrt[m]{\alpha})/K$ is unramified at $\mathfrak{p}$ (see Appendix A); therefore, it makes sense to talk about the Artin symbol $(\mathfrak{p}, K(\sqrt[m]{\alpha})/K)$. The m-th power residue symbol is given by

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = \sqrt[m]{\alpha}^{(\mathfrak{p},K(\sqrt[m]{\alpha})/K)-1}.$$

This definition does not depend on the choice of $\sqrt[m]{\alpha}$, as a simple computation shows. Moreover, it is an m-th root of unity.

Historically, of course, the m-th power residue symbol was defined before the Artin symbol, but as it is well-known, the Artin reciprocity law encompasses all the previous known reciprocity laws. The above approach makes the link between the two symbols clear. This being said, let us explain how to define the power symbol without the Artin map. We start with the following lemma used more than once in this thesis.

Lemma 2.7. *Let K be a number field and let m be a positive integer dividing w_K. Let $\mathfrak{p}$ be a prime ideal of K satisfying $(\mathfrak{p}, m) = 1$, then the m-roots of unity in K are all distinct modulo $\mathfrak{p}$.*

Proof:

Consider the polynomial $p(X) = X^m - 1$ and its reduction $\overline{p}(X)$ modulo $\mathfrak{p}$. The hypothesis implies that

$$(\overline{p}(X), \overline{p}'(X)) = 1,$$

in $O_K/\mathfrak{p}$; therefore, the roots of $p(X)$ are all distinct modulo $\mathfrak{p}$.

Q.E.D.

Lemma 2.8. *Let K be a number field and let m be a positive integer dividing w_K. Given $\alpha \in O_K$ satisfying $(m\alpha, \mathfrak{p}) = 1$, there exists a m-th root of unity $\zeta \in \mu_K$ such that*

$$\alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \equiv \zeta \pmod{\mathfrak{p}}.$$

Moreover, this ζ is unique by the previous lemma.

Proof:

We have

$$\left(\alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}}\right)^m \equiv 1 \pmod{\mathfrak{p}},$$

and therefore

$$\prod_{i=0}^{m-1}\left(\alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}}-\zeta_m^i\right)\equiv 0 \pmod{\mathfrak{p}},$$

where ζ_m is a primitive m-th root of unity in K. The ideal $\mathfrak{p}$ being prime, we can conclude the desired result.

Q.E.D.

Definition 2.9. Let K be a number field, m a positive integer dividing w_K, and $\alpha \in O_K$ be such that $(m\alpha, \mathfrak{p}) = 1$. The *m-th power residue symbol* $\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}$ can also be defined as being the unique m-th root of unity in K congruent to $\alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}}$ modulo $\mathfrak{p}$. Thus we have

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}\equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

The fact that the two definitions above agree is the object of the following proposition.

Proposition 2.10. *Let K be a number field, m a positive integer dividing w_K, and $\alpha \in O_K$ be such that $(m\alpha, \mathfrak{p}) = 1$. Then we have*

$$\sqrt[m]{\alpha}^{(\mathfrak{p},K(\sqrt[m]{\alpha})/K)-1}\equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

Proof:

Indeed, by definition of the Artin map, we have

$$\sqrt[m]{\alpha}^{(\mathfrak{p},K(\sqrt[m]{\alpha})/K)}\equiv \sqrt[m]{\alpha}^{\mathbb{N}(\mathfrak{p})} \pmod{\mathfrak{P}},$$

for all $\mathfrak{P}\,|\,\mathfrak{p}$. Since α is relatively prime with $\mathfrak{p}$, we get

$$\sqrt[m]{\alpha}^{(\mathfrak{p},K(\sqrt[m]{\alpha})/K)-1}\equiv \sqrt[m]{\alpha}^{\mathbb{N}(\mathfrak{p})-1}\equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{P}}.$$

This last equivalence being true for all $\mathfrak{P}\,|\,\mathfrak{p}$, we conclude that

$$\sqrt[m]{\alpha}^{(\mathfrak{p},K(\sqrt[m]{\alpha})/K)-1}\equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

Q.E.D.

Remark 2. *Taking $K = \mathbb{Q}$ and $m = 2$, we get back the Legendre symbol*

$$\left(\frac{n}{p}\right) = \left(\frac{n}{(p)}\right)_{\mathbb{Q},2},$$

for any integer n relatively prime with p.

If $\mathfrak{a} \in I_{K,mO_K}$, we define

$$\left(\frac{\alpha}{\mathfrak{a}}\right)_{K,m} = \prod_{\mathfrak{p}} \left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})},$$

whenever $(\alpha, \mathfrak{a}) = 1$. We remark that

$$\left(\frac{\alpha}{\mathfrak{a}}\right)_{K,m} = \sqrt[m]{\alpha}^{(\mathfrak{a},K(\sqrt[m]{\alpha})/K)-1},$$

since the Artin map is also defined by multiplicativity and $\mu_m \subseteq K$. Here are some properties satisfied by the power residue symbol.

Proposition 2.11. *If $(\mathfrak{p}, \alpha w_K) = 1$ and m divides w_K, then*

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = \left(\frac{\alpha}{\mathfrak{p}}\right)_{K,w_K}^{\frac{w_K}{m}}.$$

Proof:

One has

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,w_K} \equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{w_K}} \pmod{\mathfrak{p}}.$$

Raising to the w_K/m-th power, we get

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,w_K}^{\frac{w_K}{m}} \equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

The result follows from the uniqueness in Lemma 2.8.

Q.E.D.

Proposition 2.12. *Let K/k be a finite extension of number fields, m a positive integer dividing w_k and $\alpha \in k^\times$. Let $\mathfrak{p}$ be a prime ideal of k relatively prime to m and α, and suppose $\mathfrak{P}$ is any prime of K lying above $\mathfrak{p}$. Then*

$$\left(\frac{\alpha}{\mathfrak{P}}\right)_{K,m} = \left(\frac{\alpha}{N_{K/k}(\mathfrak{P})}\right)_{k,m}.$$

Proof:

Let $L = k(\sqrt[m]{\alpha})$, then $K \cdot L = K(\sqrt[m]{\alpha})$. A standard property satisfied by the Artin map (sometimes called the translation property) says

$$(\mathfrak{P}, K \cdot L/K)|_L = (N_{K/k}(\mathfrak{P}), L/k).$$

Thus

$$\left(\frac{\alpha}{\mathfrak{P}}\right)_{K,m} = \sqrt[m]{\alpha}^{(\mathfrak{P},K\cdot L/K)-1} = \sqrt[m]{\alpha}^{(N_{K/k}(\mathfrak{P}),L/k)-1} = \left(\frac{\alpha}{N_{K/k}(\mathfrak{P})}\right)_{k,m}.$$

Q.E.D.

Proposition 2.13. *Suppose that σ is a field automorphism of K, then*

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}^{\sigma} = \left(\frac{\alpha^\sigma}{\mathfrak{p}^\sigma}\right)_{K,m}$$

Proof:

Starting from the congruence

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} \equiv \alpha^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}},$$

and applying σ, we obtain

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}^{\sigma} \equiv (\alpha^\sigma)^{\frac{\mathbb{N}(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}^\sigma}.$$

From the uniqueness in Lemma 2.8, we conclude that

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}^{\sigma} = \left(\frac{\alpha^\sigma}{\mathfrak{p}^\sigma}\right)_{K,m}.$$

Q.E.D.

As its name suggests, the power residue symbol is sometime useful in order to see if an algebraic number is a m-th power in a given number field.

Lemma 2.14. *Let K be a number field, m a positive integer dividing w_K, and $\mathfrak{p}$ a* *prime ideal relatively prime to m. If $\alpha \in K^\times$ satisfies $(\alpha, \mathfrak{p}) = 1$, then*

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = 1 \text{ if and only if } \alpha \in (K_{\mathfrak{p}}^\times)^m.$$

Proof:

Consider the extension of local fields $K_{\mathfrak{p}}(\sqrt[m]{\alpha})/K_{\mathfrak{p}}$, where $\sqrt[m]{\alpha}$ is any m-th root of α. It is an unramified abelian extension; its Galois group is cyclic generated by the Frobenius automorphism. Now

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = \sqrt[m]{\alpha}^{\sigma_{\mathfrak{p}}-1} = 1$$

if and only if

$$\sqrt[m]{\alpha}^{\sigma_{\mathfrak{p}}} = \sqrt[m]{\alpha}.$$

This last line is equivalent to the statement $\alpha \in (K_{\mathfrak{p}}^\times)^m$.

Q.E.D.

This last lemma gives only local information. In order to get global information, we need Hasse's principle for powers:

Theorem 2.15. *Let K be a number field and m be a positive integer. Let also S be a set of primes having density 1. Then the map*

$$K^\times/(K^\times)^m \longrightarrow \prod_{v \in S} K_v^\times/(K_v^\times)^m$$

is injective, except in the following special case:

(1) *$m = 2^r m'$, with m' odd and $r \geq 3$,*

(2) *$K(\mu_{2^r})/K$ is not cyclic,*

(3) *All primes $\mathfrak{p} \in S$ dividing 2 splits in $K(\mu_{2^r})/K$.*

Proof:

For a proof, we refer to [36] page 530.

Q.E.D.

2.1.3 Stickelberger's Theorem

We are in the position to introduce the special Gauss sums generalizing the one used in the proof of the quadratic reciprocity law. As before, let m be a positive integer and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\zeta_m)$ relatively prime to m. Let $F_{\mathfrak{p}}$ be the finite field $\mathbb{Z}[\zeta_m]/\mathfrak{p}$.

Definition 2.16. We shall denote by $\chi_{\mathfrak{p}}$ the multiplicative character of $F_{\mathfrak{p}}$ defined by

$$\chi_{\mathfrak{p}}(\alpha + \mathfrak{p}) = \left(\frac{\alpha}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m},$$

whenever $\alpha + \mathfrak{p} \in F_{\mathfrak{p}}^{\times}$ (in which case $(\alpha, \mathfrak{p}) = 1$).

Remark 3. *One has to check that* $\chi_{\mathfrak{p}}(\alpha \cdot \beta + \mathfrak{p}) = \chi_{\mathfrak{p}}(\alpha + \mathfrak{p}) \cdot \chi_{\mathfrak{p}}(\beta + \mathfrak{p})$*, but this is straightforward from the definition of the* m*-th power residue symbol.*

Since the values of $\chi_{\mathfrak{p}}$ are m-th roots of unity, the Gauss sum $G(\chi_{\mathfrak{p}})$ lies in $\mathbb{Q}(\zeta_{mp})$, where p is the rational prime lying below $\mathfrak{p}$. From now on, we shall denote $G(\chi_{\mathfrak{p}})$ by $g_m(\mathfrak{p})$.

Theorem 2.17. *Let* t *be an integer satisfying* $(t, m) = 1$ *and suppose that* $t \equiv 1 \pmod{p}$*. Then*

$$g_m(\mathfrak{p})^{\sigma_t} = g_m(\mathfrak{p}^{\sigma_t}).$$

Proof:

Given $\sigma \in G_m$, we shall denote the induced field automorphism

$$F_{\mathfrak{p}} \longrightarrow F_{\mathfrak{p}^{\sigma}},$$

by $\tilde{\sigma}$ and the trace from $F_{\mathfrak{p}^{\sigma}}$ down to $\mathbb{F}_p$ by Tr'. We have

$$\begin{aligned} G(\chi_{\mathfrak{p}})^{\sigma_t} &= \sum_{\alpha \in F_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}(\alpha)^{\sigma_t} \zeta_p^{\mathrm{Tr}(\alpha)} \\ &= \sum_{\alpha \in F_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}^{\sigma_t}}(\alpha^{\tilde{\sigma}_t}) \zeta_p^{\mathrm{Tr}(\alpha)} \\ &= \sum_{\alpha \in F_{\mathfrak{p}^{\sigma_t}}^{\times}} \chi_{\mathfrak{p}^{\sigma_t}}(\alpha) \zeta_p^{\tilde{\sigma}_t^{-1}\mathrm{Tr}'(\alpha)} \end{aligned}$$

Further

$$\sum_{\alpha \in F^{\times}_{\mathfrak{p}^{\sigma_t}}} \chi_{\mathfrak{p}^{\sigma_t}}(\alpha)\zeta_p^{\tilde{\sigma}_t^{-1}\mathrm{Tr}'(\alpha)} = \sum_{\alpha \in F^{\times}_{\mathfrak{p}^{\sigma_t}}} \chi_{\mathfrak{p}^{\sigma_t}}(\alpha)\zeta_p^{\mathrm{Tr}'(\alpha)}$$
$$= G(\chi_{\mathfrak{p}^{\sigma_t}}),$$

where we used Proposition 2.13.

Q.E.D.

Given a character χ with values in μ_m, we necessarily have

$$G(\chi)^m \in \mathbb{Q}(\zeta_m).$$

Indeed, let $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q}(\zeta_m))$, that is $t \equiv 1 \pmod{m}$. Then, by Theorem 2.4, we know that

$$G(\chi)^{\sigma_t} = \overline{\chi(t)}G(\chi),$$

and since $\overline{\chi(t)} \in \mu_m$, we conclude that $G(\chi)^m$ is fixed by $\mathrm{Gal}(\mathbb{Q}(\zeta_{pm})/\mathbb{Q}(\zeta_m)$. On the other hand, Theorem 2.5 implies that the only primes of $\mathbb{Q}(\zeta_m)$ dividing the algebraic integer $G(\chi)^m$ are among the primes lying above p. Following Weil, we set

$$\Phi_m(\mathfrak{p}) = g_m(\mathfrak{p})^m.$$

Stickelberger's Theorem gives the precise decomposition of $\Phi_m(\mathfrak{p})$. One can find a proof in [64] for instance. Before we state it, here is a reciprocity law satisfied by $\Phi_m(\mathfrak{p})$ which can also be found in [64].

Proposition 2.18. *We have the following reciprocity law*

$$\left(\frac{\Phi_m(\mathfrak{p})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{-1}_{\mathbb{Q}(\zeta_m),m},$$

where $\mathfrak{q}$ *is any prime ideal of* $\mathbb{Q}(\zeta_m)$ *relatively prime to* m *and* p*. Moreover, if* $\sigma_t \in G_m$ *then*

$$\left(\frac{\Phi_m(\mathfrak{p}^{\sigma_t})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{-t}_{\mathbb{Q}(\zeta_m),m}.$$

Proof:

On one hand

$$g_m(\mathfrak{p})^{\mathbb{N}(\mathfrak{q})-1} = \Phi_m(\mathfrak{p})^{\frac{\mathbb{N}(\mathfrak{q})-1}{m}}$$
$$\equiv \left(\frac{\Phi_m(\mathfrak{p})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} \pmod{\mathfrak{q}}.$$

On the other hand

$$g_m(\mathfrak{p})^{\mathbb{N}(\mathfrak{q})} = \left(\sum_{x\in F_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}(x)\zeta_p^{\mathrm{Tr}(x)}\right)^{\mathbb{N}(\mathfrak{q})}$$
$$\equiv \sum_{x\in F_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}(x)\zeta_p^{\mathbb{N}(\mathfrak{q})\mathrm{Tr}(x)} \pmod{\mathfrak{q}}$$
$$\equiv \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-1} g_m(\mathfrak{p}) \pmod{\mathfrak{q}},$$

and therefore

$$g_m(\mathfrak{p})^{\mathbb{N}(\mathfrak{q})-1} \equiv \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-1} \pmod{q}.$$

Combining those two calculations, we get the desired equality

$$\left(\frac{\Phi_m(\mathfrak{p})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-1}.$$

For the second part, we have

$$\left(\frac{\Phi_m(\mathfrak{p}^{\sigma_t})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}^{\sigma_t}}\right)_{\mathbb{Q}(\zeta_m),m}^{-1}$$
$$= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-\sigma_t}$$
$$= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t}.$$

Q.E.D.

We remark that if $m = 2$, then

$$\Phi_2((p)) = p^*,$$

since $g_2((p))$ is nothing else than the Gauss sum used in the proof of the quadratic reciprocity law. Moreover, when $m = 2$, Proposition 2.18 gives back the quadratic reciprocity law.

We remind the reader that $\mathbb{Q}(\zeta_m)$ is a CM-field; thus, the complex conjugation induces a unique automorphism of $\mathbb{Q}(\zeta_m)$ denoted by j.

Theorem 2.19 (Stickelberger)**.** *Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field and let $\mathfrak{p}$ be a fractional ideal relatively prime to m. Then*

$$(\Phi_m(\mathfrak{p})) = \mathfrak{p}^{j\cdot\theta},$$

where

$$\theta = \sum_{\substack{t=1 \\ (t,m)=1}}^{m-1} t\sigma_t^{-1} \in \mathbb{Z}[G_m].$$

In the literature, Stickelberger's theorem is often stated as

$$(\overline{\Phi_m(\mathfrak{p})}) = (\Phi_m(\mathfrak{p}^j)) = \mathfrak{p}^{\theta}.$$

The two formulations are equivalent by Theorem 2.17.

In the next chapter, we will see that Stickelberger's Theorem is the starting point of the Brumer and the Brumer-Stark conjectures.

2.2 Dirichlet L-functions

If K is a number field, the Dedekind zeta function of a complex variable s is defined as

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathbb{N}(\mathfrak{p})}\right)^{-1} = \sum_{\mathfrak{a}} \frac{1}{\mathbb{N}(\mathfrak{a})^s},$$

where the product is taken over all prime ideals $\mathfrak{p}$ and the sum over all integral ideals $\mathfrak{a}$. This formula defines an holomorphic function on the half-plane $\mathrm{Re}(s) > 1$.

If we are given a finite abelian extension of number fields K/k with Galois group G, we can associate to each character $\chi \in \widehat{G}$ a primitive L-function as follows. Given a character $\chi \in \widehat{G}$, one says that χ is unramified at $\mathfrak{p}$ if $I_{\mathfrak{p}} \subseteq \mathrm{Ker}(\chi)$ where $I_{\mathfrak{p}}$ is the inertia group associated to $\mathfrak{p}$. Otherwise, χ is said to be ramified at $\mathfrak{p}$. If χ is unramified at $\mathfrak{p}$, then χ induces a character on $G/I_{\mathfrak{p}}$ which is the Galois group of $K^{I_{\mathfrak{p}}}/k$ and we define $\sigma_{\mathfrak{p}}$ to be the Frobenius automorphism associated to $\mathfrak{p}$ in $K^{I_{\mathfrak{p}}}/k$ (this makes sense since $\mathfrak{p}$ is unramified in $K^{I_{\mathfrak{p}}}$). The primitive L-function associated to

χ is defined as

$$L_{K/k}(s,\chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1},$$

where the product is taken over all prime ideals $\mathfrak{p}$ such that χ is unramified at $\mathfrak{p}$. Note that the trivial character χ_1 is unramified everywhere and $L_{K/k}(s,\chi_1) = \zeta_k(s)$. All these functions converge on the half-plane $\mathrm{Re}(s) > 1$, can be extended to a meromorphic function on the whole complex plane, and satisfy a functional equation. In fact, the L-functions associated to non-trivial characters are known to be holomorphic on the whole complex plane. We refer to [58] and [63] for these matters.

The Dedekind zeta function of K can be written as a product of finitely many primitive L-functions

$$\zeta_K(s) = \prod_{\chi} L_{K/k}(s,\chi), \tag{2.2}$$

where the product is over all characters χ of G. This last formula is a consequence of the following proposition whose proof is left to the reader.

Proposition 2.20. *Let K/k be an abelian extension of number fields with Galois group G. Given any prime ideal $\mathfrak{p}$ of k, one has*

$$\left(1 - \frac{1}{\mathbb{N}(\mathfrak{p})^{f_{\mathfrak{p}} s}}\right)^{-r_{\mathfrak{p}}} = \prod_{\substack{\chi\in\widehat{G} \\ \textit{unramified} \\ \textit{at } \mathfrak{p}}} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1},$$

where $f_{\mathfrak{p}}$ is the inertia index of $\mathfrak{p}$, and $r_{\mathfrak{p}}$ is the number of primes lying above $\mathfrak{p}$.

Formula (2.2) can be deduced from this last proposition, since it is equivalent to showing the following equality

$$\begin{aligned}
\zeta_K(s) &= \prod_{\chi\in\widehat{G}} \prod_{\substack{\mathfrak{p} \\ \chi \text{ unramified} \\ \text{at } \mathfrak{p}}} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1} \\
&= \prod_{\mathfrak{p}} \prod_{\substack{\chi\in\widehat{G} \\ \text{unramified} \\ \text{at } \mathfrak{p}}} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1},
\end{aligned}$$

and because the Dedekind zeta function can be rewritten as

$$\zeta_K(s) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{\mathbb{N}(\mathfrak{P})^s}\right)^{-1} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathbb{N}(\mathfrak{p})^{f_{\mathfrak{p}} s}}\right)^{-r_{\mathfrak{p}}}.$$

Therefore, we just have to show the equality of the corresponding Euler factors at $\mathfrak{p}$ and this is the content of Proposition 2.20.

This formula has several consequences. For instance, if one is interested in computing the class number of a number field, one can use this decomposition of the Dedekind zeta function with the following classical theorem in order to get an explicit way of computing the class number.

Theorem 2.21. *The Dedekind zeta function ζ_K can be extended to a meromorphic function on the whole complex plane with only one pole at $s = 1$ and this pole is a simple one. The residue at $s = 1$ is*

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{w_K\sqrt{|\Delta_K|}} \cdot h_K.$$

Using formula (2.2), we get

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \mathrm{Res}_{s=1}\,\zeta_k(s) \cdot \prod_{\chi\neq\chi_1} L_{K/k}(1,\chi).$$

Hence, knowing enough information about the base field k and also knowing the special values $L_{K/k}(1,\chi)$ for $\chi \neq \chi_1$, one can, in theory, compute the invariants of the top field. This is easier said than done, since invariants are not always easy to compute, but in some special instances, it gives beautiful results as in the quadratic case. Because of that, it has always been interesting to know the value $L_{K/k}(1,\chi)$ for $\chi \neq \chi_1$. We refer to [26] and [35] for a sample of these beautiful formulae. When the base field is $\mathbb{Q}$, such a formula is the content of the next theorem. A convenient reference for this is [60] page 37.

Theorem 2.22. *Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field and suppose that χ is a non-trivial character of G_m with conductor f. Then the associated primitive L-function satisfies*

$$L_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{f}\sum_{\substack{a=1\\(a,f)=1}}^{f}\overline{\chi(a)}\log(1-\zeta_f^a), \tag{2.3}$$

where

$$\tau(\chi) = \sum_{a=1}^{f}\chi(a)\zeta_f^a,$$

and where we take the principal branch of the logarithm.

Corollary 2.23. *Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field and suppose that χ is a non-trivial even character of G_m with conductor f. For the primitive L-function associated to χ, we have the formula*

$$L_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1,\chi) = -\frac{\tau(\chi)}{f}\sum_{\sigma\in G_f^+}\overline{\chi(\sigma)}\log c_f^{\sigma},$$

where

$$c_f = (1-\zeta_f)(1-\zeta_f^{-1}).$$

Proof:

Let χ be a non-trivial even character of G_m and let H be a complete set of representatives of $(\mathbb{Z}/f\mathbb{Z})^{\times}/\{\pm 1\}$. Starting with equation (2.3), we have

$$\begin{aligned} L_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1,\chi) &= -\frac{\tau(\chi)}{f}\sum_{a\in H}\left(\overline{\chi(a)}\log(1-\zeta_f^a)+\overline{\chi(-a)}\log(1-\zeta_f^{-a})\right)\\ &= -\frac{\tau(\chi)}{f}\sum_{a\in H}\overline{\chi(a)}\log(1-\zeta_f^a)(1-\zeta_f^{-a})\\ &= -\frac{\tau(\chi)}{f}\sum_{\sigma\in G_f^+}\overline{\chi(\sigma)}\log c_f^{\sigma}, \end{aligned}$$

after we identify G_f with $(\mathbb{Z}/f\mathbb{Z})^{\times}$ in the usual way.

Q.E.D.

This last formula could be a starting point for the abelian rank one Stark conjecture when the prime which splits completely is infinite. We will come back to this comment in the next section.

Before going any further, we remark with Stark that, using the functional equation, every result at $s=1$ can be translated to a result at $s=0$, where the formulae become simpler. For example, we have

Theorem 2.24. *The Dedekind zeta function ζ_K has a zero of order r_1+r_2-1 at $s=0$, and its Taylor expansion starts as follows:*

$$\zeta_K(s) = -\frac{h_K R_K}{w_K}s^{r_1+r_2-1}+\dots.$$

Further, for Dirichlet L-functions associated to even characters, we have

Theorem 2.25. *Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field and suppose that χ is a non-trivial even character of G_m with conductor f. Then the associated primitive L-function has a zero of order 1 at $s = 0$ and we have*

$$L'_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G_f^+} \chi(\sigma) \log c_f^{\sigma}, \tag{2.4}$$

where

$$c_f = (1 - \zeta_f)(1 - \zeta_f^{-1}).$$

The proofs of the last two theorems are direct consequences of the functional equation and will not be repeated here. See [58] again.

The element $c_m = (1 - \zeta_m)(1 - \zeta_m^{-1}) \in \mathbb{Q}(\zeta_m)^+$ is a special algebraic number. Among all its properties, we state the following relevant one.

Theorem 2.26. *If there are at least two different primes dividing m, then c_m is a unit. Otherwise, $m = p^a$ for some prime p and c_m is a p-unit.*

Proof:

If $m = p^a$ for some $a \geq 1$, then it is known that p is totally ramified in $\mathbb{Q}(\zeta_{p^a})$ and the only prime ideal lying above p is precisely $(1 - \zeta_{p^a})$. This is just another way of saying that c_m is a p-unit.

On the other hand, if m is divisible by at least 2 different primes, then starting with the equation

$$1 + X + \ldots + X^{m-1} = \prod_{i=1}^{m-1} (X - \zeta_m^i),$$

one obtains, after evaluating this last expression at $X = 1$, that

$$m = \prod_{i=1}^{m-1} (1 - \zeta_m^i). \tag{2.5}$$

Suppose that p^a divides precisely m, then one sees that the product above contains

$$p^a = \prod_{i=1}^{p^a - 1} (1 - \zeta_{p^a}^i).$$

One can remove the corresponding factor on each side of equation (2.5) and repeat the same procedure for all prime numbers dividing m. We get that 1 is equal to a factor of the product appearing in equation (2.5). Noting that every removed factor is of the form $(1 - \zeta)$, where ζ is a power-of-a-prime-th root of unity, and since m is not, we conclude that $1 - \zeta_m$ appears in this factor; therefore, it is a unit.

Q.E.D.

Chapter 3

The abelian rank one Stark conjecture, the Brumer-Stark conjecture and the Gross conjecture

In this chapter, we explain how the Brumer and the abelian rank one Stark conjecture are attempts to generalize the results of the previous chapter to any finite abelian extension of number fields. Different formulations of these conjectures are given and Gross's refinement of the (S,T)-version of the abelian rank one Stark conjecture is stated.

3.1 The Brumer conjecture

Given a number field K, it is well-known that there exist infinitely many prime ideals in every ideal class of Cl_K. Another way of looking at Stickelberger's Theorem is thus the following:

$$\theta = \sum_{\substack{t=1 \\ (t,m)=1}}^{m-1} t\sigma_t^{-1} \in \mathrm{Ann}_{\mathbb{Z}[G_m]} Cl_{\mathbb{Q}(\zeta_m)}.$$

Brumer was the first to propose how to generalize this statement to any finite abelian extension of number fields. Let m be a positive integer and let $S = \{\infty, p \,|\, m\}$.

Let us start with the partial zeta function

$$\zeta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(s,\sigma_t) = \sum_{\substack{n=1 \\ n\equiv t \pmod m}}^{\infty} \frac{1}{n^s}.$$

It can be rewritten as follows

$$\zeta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(s,\sigma_t) = \sum_{k=0}^{\infty} \frac{1}{(t+km)^s} = \frac{1}{m^s}\zeta(s,t/m),$$

where

$$\zeta(s,b) = \sum_{k=0}^{\infty} \frac{1}{(b+k)^s},$$

is the Hurwitz zeta function associated to $b \in (0,1]$. This function can be extended to a meromorphic function on the whole complex plane with a simple pole at $s = 1$. Moreover, one has

$$\zeta(0,b) = \frac{1}{2} - b.$$

One can consult Chapter 4 of [60] for these facts. Let us introduce the following function $\theta : \mathbb{C} \to \mathbb{C}[G_m]$ defined by

$$\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(s) = \sum_{\substack{t=1 \\ (t,m)=1}}^{m-1} \zeta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(s,\sigma_t)\sigma_t^{-1}.$$

Stickelberger's Theorem can be reinterpreted as

$$w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0) \in \mathrm{Ann}_{\mathbb{Z}[G_m]} Cl_{\mathbb{Q}(\zeta_m)}.$$

More precisely, given a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\zeta_m)$ relatively prime to m,

$$\mathfrak{p}^{w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0)} = (\tau(\mathfrak{p})^{w_{\mathbb{Q}(\zeta_m)}}), \tag{3.1}$$

where

$$\tau(\mathfrak{p}) = \frac{g_m(\mathfrak{p})}{(p^*)^{f/2}},$$

is a normalized Gauss sum (normalized because its absolute value is 1), f is the inertia index of p in $\mathbb{Q}(\zeta_m)$ (where $\mathfrak{p} \mid p$ as usual), and $p^* = \left(\frac{-1}{p}\right) p$. Indeed, since

$$w_{\mathbb{Q}(\zeta_m)} \theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0) = w_{\mathbb{Q}(\zeta_m)} \sum_{\substack{t=1 \\ (t,m)=1}}^{m-1} \left(\frac{1}{2} - \frac{t}{m}\right) \sigma_t^{-1}, \tag{3.2}$$

we have

$$\begin{aligned} \mathfrak{p}^{w_{\mathbb{Q}(\zeta_m)} \theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0)} &= \left(\left(\frac{p^{f/2}}{G(\overline{\chi_{\mathfrak{p}}})} \right)^{w_{\mathbb{Q}(\zeta_m)}} \right) \\ &= \left(\left(\frac{g_m(\mathfrak{p})}{p^{f/2}} \right)^{w_{\mathbb{Q}(\zeta_m)}} \right). \end{aligned}$$

This last equality is true because of Theorems 2.5 and 2.6. Finally, since $w_{\mathbb{Q}(\zeta_m)}$ is even, we can replace p by p^* in the denominator. It is useful to introduce p^* instead of p, since $\sqrt{p^*} \in \mathbb{Q}(\zeta_p)$, whereas $\sqrt{p}$ is not always in $\mathbb{Q}(\zeta_p)$; therefore, $\tau(\mathfrak{p}) \in \mathbb{Q}(\zeta_{pm})$.

We can now explain what Brumer conjectured. Let K/k be an abelian extension of number fields with Galois group G and let S be a finite set of primes of k containing $S(K/k)$. For every $\sigma \in G$, one considers the partial zeta function

$$\zeta_{K/k,S}(s,\sigma) = \sum_{\substack{(\mathfrak{a},S)=1 \\ (K/k,\mathfrak{a})=\sigma}} \frac{1}{\mathbb{N}(\mathfrak{a})^s},$$

where the sum is over all integral ideals relatively prime with the finite primes in S such that the Artin symbol satisfies $(\mathfrak{a}, K/k) = \sigma$. Then one introduces the Stickelberger function

$$\theta_{K/k,S}(s) = \sum_{\sigma \in G} \zeta_{K/k,S}(s,\sigma) \cdot \sigma^{-1}.$$

The Stickelberger function is also called the equivariant L-function for a reason which should become clear after Lemma 3.16 below. Note that in the case of $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, we had $w_{\mathbb{Q}(\zeta_m)} \theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0) \in \mathbb{Z}[G_m]$, where $S = \{\infty, p \mid m\} \supseteq S(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. This result is in fact true for any abelian extension of number fields, and is due to Deligne-Ribet ([14]) and independently to Cassou-Noguès ([3]). See also the recent work of Charollois and Dasgupta [5].

Theorem 3.1 (Deligne-Ribet, Cassou-Noguès)**.** *Given any abelian extension of number fields K/k with Galois group G and a finite set S of primes of k containing $S(K/k)$, we have*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) \cdot \theta_{K/k,S}(0) \subseteq \mathbb{Z}[G].$$

As a special case, we have

$$w_K \theta_{K/k,S}(0) \in \mathbb{Z}[G].$$

Conjecture 3.2 (Brumer)**.** *With the same notation as above, one has*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) \cdot \theta_{K/k,S}(0) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]} Cl_K.$$

In particular, this conjecture predicts

$$w_K \cdot \theta_{K/k,S}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_K.$$

3.2 The abelian rank one Stark conjecture

The abelian rank one Stark conjecture is an attempt to generalize formula (2.4) modulo some details we now make clear. First of all, it concerns S-imprimitive L-functions associated to a finite set S of primes of k containing $S(K/k)$ rather than primitive L-functions. The S-imprimitive L-function is obtained from the primitive one by removing the Euler factors at primes in S, that is

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p} \notin S} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1},$$

where we use $\sigma_{\mathfrak{p}}$ to denote the Frobenius automorphism associated to $\mathfrak{p}$. Similarly, the S-imprimitive Dedekind zeta function of a number field K is defined as

$$\zeta_{K,S}(s) = \sum_{(\mathfrak{a},S)=1} \frac{1}{\mathbb{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p} \notin S} \left(1 - \frac{1}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1}.$$

Both infinite products define an holomorphic function on the half-plane $\mathrm{Re}(s) > 1$.

Proposition 2.20 implies that the S-imprimitive Dedekind zeta function and S-imprimitive L-functions are still related to each other by the formula

$$\zeta_{K,S}(s) = \prod_{\chi} L_{K/k,S}(s,\chi), \tag{3.3}$$

where the product is over all characters of the abelian Galois group of K/k.

Theorem 3.3. *The S-imprimitive Dedekind zeta function $\zeta_{K,S}$ can be extended to a* *meromorphic function on the whole complex plane with only one pole at $s = 1$ and this pole is a simple one. Moreover, the Taylor expansion of $\zeta_{K,S}$ at $s = 0$ starts as follows:*

$$\zeta_{K,S}(s) = -\frac{h_{K,S} R_{K,S}}{w_K} s^{|S|-1} + \ldots$$

This last theorem can be deduced from Theorem 2.24 and the following simple proposition whose proof can be found in [12] page 10.

Proposition 3.4. *Let K be a number field, S a finite set of primes of k, $\mathfrak{p}$ a prime ideal not contained in S, and $S' = S \cup \{\mathfrak{p}\}$. Let also m be the order of $\mathfrak{p}$ in $Cl_{K,S}$. Then*

(1) $h_{K,S'} = h_{K,S}/m$,

(2) $R_{K,S'} = m\,(\log \mathbb{N}(\mathfrak{p}))\, R_{K,S}$.

If the Taylor expansion of $L_{K/k,S}(s,\chi)$ at $s = 0$ starts as

$$L_{K/k,S}(s,\chi) = a_S(\chi) s^{r_S(\chi)} + \ldots,$$

we should have

$$-\frac{h_{K,S} R_{K,S}}{w_K} = \prod_{\chi \in \widehat{G}} a_S(\chi), \tag{3.4}$$

and

$$|S| - 1 = \sum_{\chi \in \widehat{G}} r_S(\chi).$$

The order of vanishing $r_S(\chi)$ is known due to the following theorem.

Theorem 3.5. *Let K/k be an abelian extension of number fields and S a finite set of primes of k containing the infinite ones. Then*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \begin{cases} |S| - 1, & \textit{if } \chi = \chi_1, \\ |\{v \in S \mid G_v \subseteq \mathrm{Ker}(\chi)\}|, & \textit{if } \chi \neq \chi_1. \end{cases}$$

Proof:

See [58] page 24, Proposition 3.4.

Q.E.D.

Loosely speaking, Stark's conjecture is an attempt to describe precisely what $a_S(\chi)$ is in an equivariant way. This would give a way to break up the invariantsof K into χ-component. The following theorem could give hope that there is a way to break up the regulator into χ-components. A proof can be found in [58] (Proposition 3.4 of [58]).

Theorem 3.6. *We have the following formula for the order of vanishing of the L-functions:*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \dim_{\mathbb{C}}(\mathbb{C}E_{K,S})^{\chi},$$

where $(\mathbb{C}E_{K,S})^{\chi}$ *denotes the* χ*-component of the representation* $\mathbb{C}E_{K,S}$*.*

In the early 80s, Stark proposed the following conjecture.

Conjecture 3.7 (Stark)**.** *Let* K/k *be an abelian extension of number fields with Galois group* G *and* S *a finite set of primes of* k *satisfying the following hypotheses* (St)*:*

(St_1) S *contains* $S(K/k)$*,*

(St_2) $|S| \geq 2$*,*

(St_3) S *contains at least one prime, denoted by* v_0*, which splits completely.*

Let w_0 *be a place of* K *lying above* v_0*. Then, there exists a* S_K*-unit* $\varepsilon \in E_{K,S}$ *satisfying*

$$L'_{K/k,S}(0,\chi) = -\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^{\sigma}|_{w_0}, \tag{3.5}$$

for all characters χ *and such that*

(1) *If* $|S| = 3$*, then* $|\varepsilon|_w = 1$ *for all* $w \in S_K$ *not lying above* v_0*,*

(2) *If* $|S| = 2$*, and* $w \in S_K$ *is a place not lying above* v_0*, then* $|\varepsilon|_{w^{\sigma}} = |\varepsilon|_w$ *for all* $\sigma \in G$*.*

Moreover, the extension $K(\varepsilon^{1/w_K})/k$ *is abelian.*

Before going further, we make some remarks.

(1) The first two hypotheses guarantee that all the L-functions have order of vanishing greater than or equal to one because of Theorem 3.5 (The extended abelian rank one Stark conjecture, which we study later, removes the third condition on S).

(2) An S_K-unit satisfying the conditions above is called a Stark unit.

(3) We shall often denote this conjecture by $St(K/k, S, v_0)$ (where v_0 is the distinguished place which splits completely) or $St(K/k, S, v_0, w_0)$ if we want to emphasize the choice of the place w_0 lying above v_0.

(4) The truth of the conjecture does not depend on the choice of w_0. Indeed, if w_0' is another place lying above v_0 there exists $\tau \in G$ such that $w_0^\tau = w_0'$. Then, we have

$$\begin{aligned} -\frac{1}{w_K} \cdot \sum_{\sigma \in G} \chi(\sigma) \log |(\varepsilon^\tau)^\sigma|_{w_0'} &= -\frac{1}{w_K} \cdot \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^\sigma|_{w_0'^{\tau^{-1}}} \\ &= -\frac{1}{w_K} \cdot \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^\sigma|_{w_0} \\ &= L'_{K/k,S}(0, \chi), \end{aligned}$$

and therefore ε^τ is a Stark unit for $St(K/k, S, v_0, w_0')$.

(5) Given an abelian extension K/k and an algebraic number $\alpha \in K$, we shall say that α is w_K-abelian over k if the extension $K(\alpha^{1/w_K})/k$ is abelian.

(6) Equation (3.5) can be reinterpreted in terms of partial zeta functions instead of L-functions. First, we need a lemma.

Lemma 3.8. *Let K/k be an abelian extension of number fields with Galois group G and S a finite set of primes containing $S(K/k)$. Then*

$$L_{K/k,S}(s, \chi) = \sum_{\sigma} \chi(\sigma) \zeta_{K/k,S}(s, \sigma),$$

and

$$\zeta_{K/k,S}(s, \sigma) = \frac{1}{|G|} \sum_{\chi} \overline{\chi(\sigma)} L_{K/k,S}(s, \chi).$$

Proof:

The first formula follows from

$$\begin{aligned} L_{K/k,S}(s, \chi) &= \sum_{(\mathfrak{a}, S)=1} \frac{\chi(\sigma_{\mathfrak{a}})}{\mathbb{N}(\mathfrak{a})^s} \\ &= \sum_{\sigma \in G} \chi(\sigma) \sum_{\substack{(\mathfrak{a}, S)=1 \\ \sigma_{\mathfrak{a}} = \sigma}} \frac{1}{\mathbb{N}(\mathfrak{a})^s} \\ &= \sum_{\sigma \in G} \chi(\sigma) \zeta_{K/k,S}(s, \sigma). \end{aligned}$$

For the second formula, let $\tau \in G$ be a fixed Galois automorphism. Starting with the previous formula, multiplying both sides by $\chi(\tau^{-1})$, and summing over all characters χ of G, we get

$$\sum_{\chi \in \widehat{G}} \chi(\tau^{-1}) L_{K/k,S}(s,\chi) = \sum_{\sigma \in G} \sum_{\chi \in \widehat{G}} \chi(\sigma\tau^{-1}) \zeta_{K/k,S}(s,\sigma).$$

From the orthogonality relations, we obtain

$$\zeta_{K/k,S}(s,\tau) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \overline{\chi(\tau)} L_{K/k,S}(s,\chi),$$

and this is what we wanted to show.

Q.E.D.

Using this lemma, it is simple to verify that equation (3.5) is equivalent to saying

$$-w_K \zeta'_{K/k,S}(0,\sigma) = \log |\varepsilon^{\sigma}|_{w_0}, \tag{3.6}$$

for all $\sigma \in G$.

(7) The specification of the absolute values of a Stark unit in the abelian rank one Stark conjecture and formula (3.6) determine all its absolute values. Therefore, the uniqueness of a Stark unit up to a root of unity follows from the following theorem (and the product formula in the case $|S| = 2$) which is due to Kronecker (see Lemma 1.6 page 4 of [60] for a proof).

Theorem 3.9. *Suppose that α is an algebraic number in a number field K. If α satisfies $|\alpha|_v = 1$ at every place v of K, then α is necessarily a root of unity.*

(8) In order to prove the abelian condition in specific cases, it is often useful to use Theorem A.5.

(9) Even though the abelian rank one Stark conjecture is stated for any finite set S of primes of k containing $S(K/k)$, the goal is to prove it for minimal sets S because of the following theorem.

Theorem 3.10. *If $St(K/k,S,v_0)$ is true and $S \subseteq S'$, then $St(K/k,S',v_0)$ is also true. Moreover, the corresponding Stark units ε_S and $\varepsilon_{S'}$ are related through the* *following formula*

$$\varepsilon_{S'} = \varepsilon_S^{\alpha},$$

where

$$\alpha = \prod_{v \in S' \smallsetminus S} (1 - \sigma_v^{-1}).$$

Proof:

It suffices to show the above result when $S' = S \cup \{\mathfrak{p}\}$. We have

$$L_{K/k,S'}(s,\chi) = L_{K/k,S}(s,\chi) \cdot \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(p)^s}\right).$$

Therefore

$$L'_{K/k,S'}(0,\chi) = L'_{K/k,S}(0,\chi) \cdot (1 - \chi(\sigma_{\mathfrak{p}})),$$

since $L_{K/k,S}(0,\chi) = 0$. Thus

$$\begin{aligned} L'_{K/k,S'}(0,\chi) &= \left(-\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^{\sigma}|_{w_0}\right)\cdot(1-\chi(\sigma_{\mathfrak{p}})) \\ &= -\frac{1}{w_K}\left(\sum_{\sigma}\chi(\sigma)\log|\varepsilon^{\sigma}|_{w_0} + \sum_{\sigma}\chi(\sigma_{\mathfrak{p}}\sigma)\log|\varepsilon^{\sigma}|_{w_0}\right) \\ &= -\frac{1}{w_K}\left(\sum_{\sigma}\chi(\sigma)\log|\varepsilon^{\sigma}|_{w_0} + \sum_{\sigma}\chi(\sigma)\log|\varepsilon^{\sigma_{\mathfrak{p}}^{-1}\sigma}|_{w_0}\right) \\ &= -\frac{1}{w_K}\sum_{\sigma}\chi(\sigma)\log|(\varepsilon^{1-\sigma_{\mathfrak{p}}^{-1}})^{\sigma}|_{w_0}. \end{aligned}$$

We leave it to the reader to convince himself that $\varepsilon^{1-\sigma_{\mathfrak{p}}^{-1}}$ has the correct absolute values and it satisfies the abelian condition.

Q.E.D.

(10) If there are two different places in S which split completely in K and $|S| \geq 3$, Theorem 3.5 implies that all the L-functions have order of vanishing at least two; thus, one can take $\varepsilon = 1$ as a Stark unit.

3.3 Quadratic extensions

We recall here a result contained in [58] which we use later on in this thesis.

Theorem 3.11. *Let K/k be a quadratic extension of number fields, then $St(K/k, S, v)$ is true. Moreover, if $|S| \geq 3$, then there exists an $\eta \in E_{K,S}$ such that*

(1) $|\eta|_w = 1$ *for all* $w \in S_K$ *not lying above* v,

(2) *The extension* $K(\eta^{1/w_K})/k$ *is abelian,*

(3) *A Stark unit* ε *for* $St(K/k, S, v)$ *is given by*

$$\varepsilon = \eta^{2^{|S|-3}}.$$

Proof:

We refer to [58] page 104 for the proof of this result.

Q.E.D.

3.4 The prime which splits completely is infinite

3.4.1 The case of $\mathbb{Q}(\zeta_m)^+/\mathbb{Q}$

Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_m)^+$ be fixed, and let $S = \{\infty, p \mid m\}$. In this case, the prime v_0 which splits completely is the unique infinite place ∞ of $\mathbb{Q}$. Formula (2.4) is the first step in order to get a Stark unit. The main problem is to deduce a formula for the derivative at $s = 0$ of imprimitive L-functions. Using Theorem 3.10, we have

$$L'_{K/k,S}(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G_f^+} \chi(\sigma) \log(c_f^\alpha)^\sigma,$$

where

$$\alpha = \prod_{\substack{p \mid m \\ p \nmid f}} (1 - \sigma_p^{-1}) \in \mathbb{Z}[G_f^+].$$

In order to go further, we need the following theorem whose proof is left to the reader.

Proposition 3.12. *Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field and let*

$$c_m = (1 - \zeta_m)(1 - \zeta_m^{-1}) \in \mathbb{Q}(\zeta_m)^+.$$

Suppose $n|m$ and let σ_p denotes the Frobenius associated to p in the extension $\mathbb{Q}(\zeta_n)^+$. We have the following formula relating c_m and c_n:

$$N_{\mathbb{Q}(\zeta_m)^+/\mathbb{Q}(\zeta_n)^+}(c_m) = c_n^{\alpha},$$

where

$$\alpha = \prod_{\substack{p|m \\ p\nmid n}} (1 - \sigma_p^{-1}) \in \mathbb{Z}[G_n^+].$$

Using this last proposition and the fact that

$$\left(1 - \exp\left(\frac{2\pi it}{m}\right)\right)\left(1 - \exp\left(\frac{-2\pi it}{m}\right)\right) > 0,$$

for all $t \in \mathbb{Z}$ such that $(t, m) = 1$, we compute

$$\begin{aligned} L'_{K/k,S}(0,\chi) &= -\frac{1}{2}\sum_{\sigma\in G_f^+} \chi(\sigma)\log\left(N_{\mathbb{Q}(\zeta_m)^+/\mathbb{Q}(\zeta_f)^+}(c_m)\right)^{\sigma} \\ &= -\frac{1}{2}\sum_{\sigma\in G_m^+} \chi(\sigma)\log c_m^{\sigma} \\ &= -\frac{1}{2}\sum_{\sigma\in G_m^+} \chi(\sigma)\log |c_m^{\sigma}|_{w_0}, \end{aligned}$$

where w_0 is the infinite place corresponding to the embedding implicitly chosen by setting $\zeta_m = \exp\left(\frac{2\pi i}{m}\right)$.

We leave it to the reader to check that c_m has the correct absolute values at every place $w \in S$ not lying above ∞.

Finally, we have to check that $\mathbb{Q}(\zeta_m)^+(\sqrt{c_m})/\mathbb{Q}$ is an abelian extension, but this is clear because, since

$$c_m = -\zeta_m^{-1}(1 - \zeta_m)^2,$$

and thus $\sqrt{c_m}$ lies in some cyclotomic field.

3.4.2 The case of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, $d > 0$

In the case of a real quadratic number field, a Stark unit can be computed explicitly using the following proposition.

Proposition 3.13. *Let $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{d})$ where d is a positive square-free integer, and h_K be the class number of K. Let also $S = S(K/\mathbb{Q})$. The place ∞ is the distinguished place in S which splits completely. Fix a place w of K lying above ∞.*

(1) *Suppose that $|S| \geq 3$. If η is a fundamental unit satisfying $|\eta|_w < 1$, then a Stark unit for $St(K/\mathbb{Q}, S, \infty)$ is given by*

$$\varepsilon = \eta^{h_K}.$$

(2) *Suppose that $|S| = 2$, which is the same as saying that $K = \mathbb{Q}(\sqrt{p})$ for some prime p satisfying $p \equiv 1 \pmod 4$ and $S = \{\infty, p\}$. If η is a fundamental unit satisfying $|\eta|_w < 1$, then a Stark unit is given by*

$$\varepsilon = \eta^{h_K}\sqrt{p}.$$

Proof:

Since S contains only ramified finite primes and the infinite prime is the one splitting completely, we conclude that

$$\mathrm{ord}_{s=0} L_{K/k,S}(s, \chi) = 1,$$

where χ is the non-trivial character. Formula (3.3) says in this case the following

$$-\frac{h_{K,S}R_{K,S}}{w_K} = -\frac{h_{\mathbb{Q},S}R_{\mathbb{Q},S}}{w_{\mathbb{Q}}} L'_{K/k,S}(0, \chi).$$

Therefore, we know that

$$L'_{K/k,S}(0, \chi) = h_K R_K.$$

If $|S| \geq 3$, we have

$$L'_{K/k,S}(0, \chi_1) = 0,$$

where χ_1 is the trivial character. So first of all, for the trivial character, we have

$$
\begin{aligned}
-\frac{1}{w_K}\left(\chi_1(1)\log|\varepsilon|_w + \chi_1(\sigma)\log|\varepsilon^\sigma|_w\right) &= -\frac{1}{w_K}\log|\varepsilon\varepsilon^\sigma|_w \\
&= -\frac{1}{w_K}\log|\eta\eta^\sigma|_w^{h_K} \\
&= -\frac{h_K}{w_K}\log|N_{K/\mathbb{Q}}(\eta)|_w \\
&= 0 \\
&= L'_{K/k,S}(0,\chi_1).
\end{aligned}
$$

Further, for the non-trivial character χ, we have

$$
\begin{aligned}
-\frac{1}{w_K}\left(\chi(1)\log|\varepsilon|_w + \chi(\sigma)\log|\varepsilon^\sigma|_w\right) &= -\frac{1}{w_K}\log\left|\frac{\varepsilon}{\varepsilon^\sigma}\right|_w \\
&= -\frac{1}{w_K}\log\left|\frac{\eta}{\eta^\sigma}\right|_w^{h_K} \\
&= -\frac{h_K}{w_K}\log\left|\frac{\eta^2}{N_{K/\mathbb{Q}}(\eta)}\right|_w \\
&= -\frac{2h_K}{w_K}\log|\eta|_w \\
&= h_K R_K \\
&= L'_{K/k,S}(0,\chi).
\end{aligned}
$$

It is also clear that $|\varepsilon|_w = 1$ for all finite prime w, so we just have to check that $K(\varepsilon^{1/2})/\mathbb{Q}$ is an abelian extension. In order to do so, we use Theorem A.5. We let

$$n_1 = n_\sigma = 1, \quad \alpha_1 = 1,$$

and we have to find α_σ. We compute

$$\varepsilon^{\sigma-1} = (\eta^{h_K})^{\sigma-1} = \left(\frac{\eta^\sigma}{\eta}\right)^{h_K} = \left(\frac{N_{K/\mathbb{Q}}(\eta)}{\eta^2}\right)^{h_K}.$$

If the norm of the fundamental unit is 1, then we take

$$\alpha_\sigma = \frac{1}{\eta^{h_K}} \in K.$$

On the other hand, if the norm of the fundamental unit is -1, then we get

$$\varepsilon^{\sigma-1} = \left(\frac{-1}{\eta^2}\right)^{h_K}.$$

It is a classical result pertaining to genus theory that if the norm of the fundamental unit is -1, then

$$2^{|S|-2} \mid h_K^+ = h_K,$$

where h_K^+ is the narrow class number (see Theorem 3.14 below). Therefore we take

$$\alpha_\sigma = \frac{(-1)^{h_K/2}}{\eta^{h_K}} \in K.$$

It is then simple to check that

$$\alpha_1^{\sigma-1} = \alpha_\sigma^{1-1} = 1.$$

If $S = \{\infty, p\}$ where $p \equiv 1 \pmod 4$, then the only things which change above are

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi_1) = 1 \text{ and } L'_{K/k,S}(0,\chi_1) = -\frac{1}{2}\log p.$$

As for the abelian condition, we set

$$n_1 = n_\sigma = 1, \quad \alpha_1 = 1,$$

and we have to find α_σ. So again we compute

$$\varepsilon^{\sigma-1} = -\left(\frac{N_{K/\mathbb{Q}}(\eta)}{\eta^2}\right)^{h_K}.$$

By Corollary 3.15 below, we know that $N_{K/\mathbb{Q}}(\eta) = -1$ and h_K is odd, therefore, we get

$$\varepsilon^{\sigma-1} = \left(\frac{1}{\eta^{h_K}}\right)^2,$$

and we take $\alpha_\sigma = 1/\eta^{h_K}$. Again, it is simple to check that

$$\alpha_1^{\sigma-1} = \alpha_\sigma^{1-1} = 1.$$

Q.E.D.

Given a finite abelian group G, the p-rank of G, denoted by $r_p(G)$, is the number of cyclic direct summands of its unique p-Sylow subgroup. A simple computation shows it is also given by

$$r_p(G) = \dim_{\mathbb{F}_p}(G/pG),$$

if the operation of the group is thought of as being additive.

Theorem 3.14. *Let K be a quadratic number field and suppose that t is the number of distinct primes dividing the discriminant of K. Then*

$$r_2(Cl_K^+) = t - 1,$$

where Cl_K^+ is the narrow class group of K. Moreover, if K is a real quadratic field and η is a fundamental unit, then

(1) $h_K^+ = h_K$ *if and only if* $N_{K/\mathbb{Q}}(\eta) = -1$.

(2) $h_K^+ = 2h_K$ *if and only if* $N_{K/\mathbb{Q}}(\eta) = 1$.

Proof:

A proof can be found in [30] page 160.

Q.E.D.

Corollary 3.15. *Let K be a quadratic number field and assume there is only one prime dividing the discriminant of K, then h_K^+ (and therefore h_K) is necessarily odd. Moreover, if K is totally real, the norm of the fundamental unit is necessarily -1.*

3.5 The distinguished prime which splits completely is finite (The Brumer-Stark conjecture)

In this section, we explain what was called by Tate the Brumer-Stark conjecture. Suppose we are given a finite abelian extension of number fields K/k with Galois group G and a set S of primes of k containing $S(K/k)$. Suppose moreover that S satisfies the hypotheses (St) of the abelian rank one Stark conjecture. We also suppose that $St(K/k, S, v_0, w_0)$ is true and the distinguished prime v_0 which splits completely is finite. Because of that, we denote v_0 and w_0 by $\mathfrak{p}$ and $\mathfrak{P}$ respectively. We introduce the set $R = S \setminus \{\mathfrak{p}\}$. Note that R still contains the infinite and the ramified primes.

Starting from equation (3.6) we get

$$\begin{aligned} -w_K \zeta'_{K/k,S}(0,\sigma) &= \log |\varepsilon^\sigma|_{\mathfrak{P}} \\ &= \log \left(\mathbb{N}(\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}^{\sigma^{-1}}}(\varepsilon)} \right). \end{aligned}$$

Therefore,

$$\mathrm{ord}_{\mathfrak{P}^{\sigma^{-1}}}(\varepsilon) = w_K \frac{\zeta'_{K/k,S}(0,\sigma)}{\log \mathbb{N}(\mathfrak{P})},$$

and thus

$$(\varepsilon) = \mathfrak{P}^{w_K \frac{\theta'_{K/k,S}(0)}{\log \mathbb{N}(\mathfrak{P})}}.$$

Lemma 3.16. *If $S \supseteq S(K/k)$, then the Stickelberger function $\theta_{K/k,S}$ satisfies*

$$\theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S}(s,\chi) e_{\chi^{-1}}.$$

Proof:

This follows from Lemma 3.8. We have

$$\begin{aligned} \theta_{K/k,S}(s) &= \sum_{\sigma} \zeta_{K/k,S}(s,\sigma) \cdot \sigma^{-1} \\ &= \sum_{\sigma} \frac{1}{|G|} \sum_{\chi} \overline{\chi(\sigma)} L_{K/k,S}(s,\chi) \cdot \sigma^{-1} \\ &= \sum_{\chi} L_{K/k,S}(s,\chi) \frac{1}{|G|} \sum_{\sigma} \overline{\chi(\sigma)} \sigma^{-1} \\ &= \sum_{\chi} L_{K/k,S}(s,\chi) e_{\chi^{-1}}. \end{aligned}$$

Q.E.D.

Lemma 3.17. *We have the following formulae*

$$\theta_{K/k,S\cup\{\mathfrak{q}\}}(s) = \left(1 - \frac{\sigma_{\mathfrak{q}}^{-1}}{\mathbb{N}(\mathfrak{q})^s}\right) \theta_{K/k,S}(s),$$

and

$$\theta'_{K/k,S\cup\{\mathfrak{q}\}}(s) = \frac{\sigma_{\mathfrak{q}}^{-1}}{\mathbb{N}(\mathfrak{q})^s} \log \mathbb{N}(\mathfrak{q}) \theta_{K/k,S}(s) + \left(1 - \frac{\sigma_{\mathfrak{q}}^{-1}}{\mathbb{N}(\mathfrak{q})^s}\right) \theta'_{K/k,S}(s).$$

Proof:

The second formula is a direct consequence of the first one. The first one can be seen as follows.

$$\begin{aligned}\theta_{K/k,S\cup\{\mathfrak{q}\}}(s) &= \sum_{\chi} L_{S\cup\{\mathfrak{q}\}}(s,\chi)e_{\overline{\chi}} \\ &= \sum_{\chi} L_{K/k,S}(s,\chi)\left(1-\frac{\overline{\chi(\sigma_{\mathfrak{q}}^{-1})}}{\mathbb{N}(\mathfrak{q})^s}\right)e_{\overline{\chi}} \\ &= \left(1-\frac{\sigma_{\mathfrak{q}}^{-1}}{\mathbb{N}(\mathfrak{q})^s}\right)\theta_{K/k,S}(s).\end{aligned}$$

Q.E.D.

Corollary 3.18. *With the same notation as in the previous lemma, if* $\mathfrak{q}$ *splits completely in* K/k *then*

$$\theta_{K/k,S}(0) = \frac{\theta'_{K/k,S\cup\{\mathfrak{q}\}}(0)}{\log\mathbb{N}(\mathfrak{q})}.$$

Going back to our discussion before this last series of lemmata, we get

$$(\varepsilon) = \mathfrak{P}^{w_K\theta_{K/k,R}(0)}.$$

This last equation is reminiscent of Brumer's conjecture. Summarizing, we have the following theorem.

Theorem 3.19. *Let* K/k *be a finite abelian extension of number fields and let* R *be a finite set of prime of* k *containing* $S(K/k)$*. Suppose that* $|R| \geq 2$*. Then the abelian rank one Stark conjecture* $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ *is equivalent to the existence of an* $\varepsilon \in K^\times$ *satisfying*

(1) $w_K\theta_{K/k,R}(0) \in \mathbb{Z}[G]$ *(which we already know it is true by Theorem 3.1),*

(2) $(\varepsilon) = \mathfrak{P}^{w_K\theta_{K/k,R}(0)}$*,*

(3) ε *is an anti-unit, that is all its archimedean absolute values are one,*

(4) $K(\varepsilon^{1/w_K})/k$ *is an abelian extension.*

We can now state the Brumer-Stark conjecture.

Conjecture 3.20 (Brumer-Stark)**.** *Let* K/k *be a finite abelian extension of number fields. Let* R *a finite set of primes of* k *containing* $S(K/k)$ *and satisfying* $|R| \geq 2$*. For all fractional ideals* $\mathfrak{a}$ *of* K*, there exists* $\varepsilon_R(\mathfrak{a}) \in K^\times$ *such that*

(1) $\mathfrak{a}^{w_K \theta_{K/k,R}(0)} = (\varepsilon_R(\mathfrak{a}))$,

(2) $\varepsilon_R(\mathfrak{a})$ *is an anti-unit,*

(3) $K(\varepsilon_R(\mathfrak{a})^{1/w_K})/k$ *is an abelian extension.*

The element $\varepsilon_R(\mathfrak{a})$ is called a Brumer-Stark element at $\mathfrak{a}$. We shall denote this conjecture by $BrSt(K/k, R)$.

Remark 4. *If $\theta_{K/k,R}(0) = 0$, then the conjecture is trivial. Hence, the Brumer-Stark conjecture has some content precisely when k is totally real and K is totally complex (or k is a quadratic imaginary number field and K/k is unramified. Since the conjecture is known in that case ([53]), we can avoid to talk about it). Therefore, we can suppose that $|R| \geq 2$ and thus that a Stark unit ε for $St(K/k, S, \mathfrak{p})$ should satisfy $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above $\mathfrak{p}$. Properties (2) and (3) of Theorem 3.19 are clearly equivalent to this latter requirement on the absolute values of ε. When dealing with the Brumer-Stark conjecture it is convenient to introduce the notion of anti-unit. From now on, we shall denote the group of anti-units of a number field K by K^0; it consists of the algebraic numbers in K whose absolute values at infinite places are one.*

For future use, we also state the following two lemmata.

Lemma 3.21. *Let K/k be a finite abelian extension of number fields and R a finite set of places of k containing $S(K/k)$ and satisfying $|R| \geq 2$. Given $v \in R$, we have*

$$N_{G_v} \theta_{K/k,R}(0) = 0,$$

where

$$N_{G_v} = \sum_{\sigma \in G_v} \sigma.$$

Proof:

For a character χ of G, we let e_χ denote the usual idempotent

$$e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathbb{C}[G].$$

If χ_1 is the trivial character, then

$$e_{\chi_1} \cdot N_{G_v} \theta_{K/k,R}(0) = |G_v| \cdot \zeta_{k,R}(0) = 0,$$

since $|R| \geq 2$. If $\chi \neq \chi_1$ then there are two possibilities: Either $G_v \subseteq \mathrm{Ker}(\chi)$ or not. In the former case, $e_\chi \cdot \theta_{K/k,R}(0) = 0$ and in the latter case $e_\chi \cdot N_{G_v} = 0$ by the orthogonality relations. This is what we wanted to show.

Q.E.D.

Lemma 3.22. *Let K/k be a finite abelian extension of number fields and R a finite set of places containing $S(K/k)$. For each $\sigma \in G$, choose $n_\sigma \in \mathbb{Z}$ such that $\zeta^\sigma = \zeta^{n_\sigma}$. Let n be any integer and suppose that*

(1) $\frac{w_K \cdot \theta_{K/k,R}(0)}{n} \in \mathbb{Z}[G]$,

(2) $\frac{(\sigma - n_\sigma) \cdot \theta_{K/k,R}(0)}{n} \in \mathbb{Z}[G]$.

Then given any $\alpha \in K^\times$, the algebraic number

$$\alpha^{\frac{w_K \cdot \theta_{K/k,R}(0)}{n}} \in K^\times$$

is w_K-abelian over k.

Proof:

We use Theorem A.5. Let

$$\beta = \alpha^{\frac{w_K \cdot \theta_{K/k,R}(0)}{n}} \in K^\times,$$

and set

$$\beta_\sigma = \alpha^{\frac{(\sigma - n_\sigma) \cdot \theta_{K/k,R}(0)}{n}}.$$

We clearly have

$$\beta^{\sigma - n_\sigma} = \beta_\sigma^{w_K},$$

and

$$\beta_\sigma^{\tau - n_\tau} = \beta_\tau^{\sigma - n_\sigma},$$

for all $\sigma, \tau \in G$.

Q.E.D.

Theorem 3.23. *Let K/k be a finite abelian extension of number fields and R a finite set of primes of k containing $S(K/k)$. Suppose that $|R| \geq 2$. Then $BrSt(K/k, R)$ is equivalent to $St(K/k, R \cup \{\mathfrak{p}\}, \mathfrak{p})$, where $\mathfrak{p}$ runs over all prime ideals of k which split completely in K. Moreover, if $\varepsilon_R(\mathfrak{a})$ is a Brumer-Stark element at $\mathfrak{a}$, then a Stark unit for $St(K/k, R \cup \{\mathfrak{p}\}, \mathfrak{p}, \mathfrak{P})$ is given by $\varepsilon_R(\mathfrak{P})$.*

Proof:

This follows form Cebotarev's theorem and Lemma 3.22. See [58] page 168 and also Proposition 5.9 below.

Q.E.D.

3.5.1 The case of $\mathbb{Q}(\zeta_m)/\mathbb{Q}$

Let $\mathbb{Q}(\zeta_m)$ be a fixed cyclotomic field with conductor $\infty \cdot m$ (i.e. $m \not\equiv 2 \pmod 4$) and let $R = S(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = \{\infty, q \mid m\}$. Because of equation (3.1), we know that if $\mathfrak{p}$ lies above p, and p is relatively prime with $w_{\mathbb{Q}(\zeta_m)}$, then

$$\mathfrak{p}^{w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},R}(0)} = (\tau(\mathfrak{p})^{w_{\mathbb{Q}(\zeta_m)}}),$$

where

$$\tau(\mathfrak{p}) = \frac{g_m(\mathfrak{p})}{(p^*)^{f/2}}.$$

In order to show that

$$\varepsilon_R(\mathfrak{p}) = \tau(\mathfrak{p})^{w_{\mathbb{Q}(\zeta_m)}} \tag{3.7}$$

is a Brumer-Stark element at $\mathfrak{p}$, we have to check the two remaining conditions of $BrSt(K/k, R)$. The fact that $\mathbb{Q}(\zeta_m)(\tau(\mathfrak{p}))$ is an abelian extension of $\mathbb{Q}$ is clear because we know that $\tau(\mathfrak{p}) \in \mathbb{Q}(\zeta_{pm})$. Thus, we just have to check that it is an anti-unit, but

$$|\tau(\mathfrak{p})^{w_{\mathbb{Q}(\zeta_m)}}| = 1,$$

since $|g_m(\mathfrak{p})| = p^{f/2}$ by Theorem 2.5. Moreover, if $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, we can also suppose that $t \equiv 1 \pmod p$ since $(p, m) = 1$ and Theorem 2.17 implies that

$$g_m(\mathfrak{p})^{\sigma_t} = g_m(\mathfrak{p}^{\sigma_t}).$$

Therefore, we conclude

$$|\varepsilon_R(\mathfrak{p})^{\sigma_t}| = 1,$$

for all $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ since $|g_m(\mathfrak{p}^{\sigma_t})| = p^{f/2}$ as well. This means precisely that $\varepsilon_R(\mathfrak{p})$ is an anti-unit; thus, the Brumer-Stark conjecture is true for full cyclotomic fields. In particular, if p splits completely in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, then $St(\mathbb{Q}(\zeta_m)/\mathbb{Q}, R \cup \{p\}, p, \mathfrak{p})$ is true with a Stark unit given by $\varepsilon_R(\mathfrak{p})$.

3.5.2 The case of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, $d<0$

In the case of an imaginary quadratic number field, a Stark unit can be computed explicitly using the following proposition.

Proposition 3.24. *Let $k=\mathbb{Q}$, $K=\mathbb{Q}(\sqrt{-d})$ where d is a positive square-free integer, and let h_K be the class number of K. Let $R=S(K/\mathbb{Q})$ and consider*

$$S = R \cup \{p\},$$

where p is a finite prime which splits completely in $K/\mathbb{Q}$. If $\mathfrak{p}$ is a prime lying above p, then

$$\mathfrak{p}^{h_K} = (\pi),$$

for some $\pi \in O_K$ which we can find explicitly. We choose the place w_0 in the statement of the conjecture to be $\mathfrak{p}$. Then, a Stark unit is given by

$$\varepsilon = \frac{\pi}{\pi^{\sigma}},$$

where σ is the non-trivial Galois automorphism of $K/\mathbb{Q}$.

Proof:

Since R contains only ramified primes (finite or infinite), we conclude that

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = 1,$$

where χ is the non-trivial character. Thus, in this case, formula (3.3) says that

$$-\frac{h_{K,S}R_{K,S}}{w_K} = -\frac{h_{\mathbb{Q},S}R_{\mathbb{Q},S}}{w_{\mathbb{Q}}} L'_{K/k,S}(0,\chi).$$

Therefore, using Proposition 3.4, we have

$$L'_{K/k,S}(0,\chi) = \frac{2h_K \log p}{w_K},$$

and since $|S| \geq 3$

$$L'_{K/k,S}(0,\chi_1) = 0,$$

where χ_1 is the trivial character. So, for the trivial character χ_1, we have

$$\begin{aligned}
-\frac{1}{w_K}\left(\chi_1(1)\log|\varepsilon|_{\mathfrak{p}}+\chi_1(\sigma)\log|\varepsilon^\sigma|_{\mathfrak{p}}\right) &= -\frac{1}{w_K}\log|\varepsilon\varepsilon^\sigma|_{\mathfrak{p}} \\
&= -\frac{1}{w_K}\log\left|\frac{\pi}{\pi^\sigma}\cdot\frac{\pi^\sigma}{\pi}\right|_{\mathfrak{p}} \\
&= 0 \\
&= L'_{K/k,S}(0,\chi_1).
\end{aligned}$$

Further, for the non-trivial character χ, we have

$$\begin{aligned}
-\frac{1}{w_K}\left(\chi(1)\log|\varepsilon|_{\mathfrak{p}}+\chi(\sigma)\log|\varepsilon^\sigma|_{\mathfrak{p}}\right) &= -\frac{1}{w_K}\log\left|\frac{\varepsilon}{\varepsilon^\sigma}\right|_{\mathfrak{p}} \\
&= -\frac{1}{w_K}\log\left|\left(\frac{\pi}{\pi^\sigma}\right)^2\right|_{\mathfrak{p}} \\
&= -\frac{2}{w_K}\log\left|\frac{\pi}{\pi^\sigma}\right|_{\mathfrak{p}} \\
&= \frac{2}{w_K}\mathrm{ord}_{\mathfrak{p}}(\pi/\pi^\sigma)\log\mathbb{N}(\mathfrak{p}) \\
&= \frac{2h_K\log p}{w_K} \\
&= L'_{K/k,S}(0,\chi).
\end{aligned}$$

We must check further that $|\varepsilon|_w = 1$ for all $w\in S_K$ not dividing p. This last fact is clear if w is a finite prime. If w is an infinite prime it follows from

$$\left|\frac{\pi}{\pi^\sigma}\right| = \left|\frac{\pi^\sigma}{\pi}\right| = 1,$$

since σ is just the usual complex conjugation (K is a quadratic imaginary number field).

At last, we have to check that $K(\varepsilon^{1/w_K})/\mathbb{Q}$ is an abelian extension. In order to do this, we use Theorem A.5. Assume $w_K = 2$. Letting

$$n_1 = n_\sigma = 1,\quad \alpha_1 = 1,\quad \alpha_\sigma = \frac{\pi^\sigma}{\pi},$$

it is simple to check that all conditions of Theorem A.5 are satisfied, namely

$$\varepsilon^{1-1} = \alpha_1^2,\quad \varepsilon^{\sigma-1} = \alpha_\sigma^2,\quad \alpha_1^{\sigma-1} = \alpha_\sigma^{1-1} = 1.$$

If $w_K = 4$ or $w_K = 6$, the proof is similar and left to the reader.

Q.E.D.

For example, take the field $K = \mathbb{Q}(\sqrt{-7})$, which is of class number one, and let $S = \{\infty, 7, 37\}$, where 37 is the prime which splits completely in $K/\mathbb{Q}$. Let $\mathfrak{p}$ be a prime lying above 37. We have $\mathfrak{p} = (\pi)$ for some $\pi \in O_K$. Since $\mathfrak{p}\overline{\mathfrak{p}} = (\pi \cdot \overline{\pi})$, we can solve the following equation

$$4 \cdot 37 = a^2 + 7b^2.$$

One finds $a = 6$ and $b = 4$. Therefore, a Stark unit for these data is

$$\varepsilon = \frac{3 + 2\sqrt{-7}}{3 - 2\sqrt{-7}} = \left(\frac{3 + 2\sqrt{-7}}{\sqrt{37}}\right)^2.$$

As another example, take the field $K = \mathbb{Q}(\sqrt{-7 \cdot 37})$, which is of class number four, and let $S = \{\infty, 7, 37, 5\}$ with 5 being the prime which splits completely in $K/\mathbb{Q}$. Let $\mathfrak{p}$ be a prime lying above 5, then we have $\mathfrak{p}^4 = (\pi)$ for some π which we can find explicitly as in the previous example. We find

$$\pi = \frac{13 + 3\sqrt{-7 \cdot 37}}{2},$$

and we get that a Stark unit is given by

$$\varepsilon = \frac{13 + 3\sqrt{-7 \cdot 37}}{13 - 3\sqrt{-7 \cdot 37}} = \left(\frac{13 + 3\sqrt{-7 \cdot 37}}{50}\right)^2.$$

3.6 The case of arbitrary abelian extensions of $\mathbb{Q}$

So far, we know the abelian rank one Stark conjecture for quadratic extensions of $\mathbb{Q}$, $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, and $\mathbb{Q}(\zeta_m + \zeta_m^{-1})/\mathbb{Q}$. We also know the Brumer-Stark conjecture for full cyclotomic fields $\mathbb{Q}(\zeta_m)/\mathbb{Q}$. The goal of this section is to complete the proof of these conjectures for arbitrary abelian extensions of $\mathbb{Q}$.

Let K/k be an abelian extension of number fields and suppose S is a finite set of primes of k containing $S(K/k)$. Assume that S contains a prime which splits completely in K/k, say v_0. If M is an intermediate field, i.e. $k \subseteq M \subseteq K$, then S still satisfies the same properties for M/k.

Theorem 3.25. *Let $k \subseteq M \subseteq K$ be a tower of number fields and let S be a finite set of primes of k satisfying the hypotheses* (St) *for K/k. If $St(K/k, S, v_0)$ is true, then $St(M/k, S, v_0)$ is also true.*

Proof:

We refer again to [58], Proposition 3.5, page 92. See also Proposition 5.18 below. For future use, we remark that if $w_M = w_K$, and ε_K is a Stark unit for $St(K/k, S, v_0)$, then a Stark unit ε_M for $St(M/k, S, v_0)$ can be taken to be

$$\varepsilon_M = N_{K/M}(\varepsilon_K).$$

Q.E.D.

This theorem concludes the proof of the abelian rank one Stark conjecture for all totally real abelian extensions of $\mathbb{Q}$. Indeed, if K is such a field, then $K \subseteq \mathbb{Q}(\zeta_m)^+$ for some m, and we know that the abelian rank one Stark conjecture holds for $\mathbb{Q}(\zeta_m)^+/\mathbb{Q}$ (see Section 3.4.1).

On the other hand, when the prime splitting completely is finite (i.e. K is a CM-field) the following possibility can occur. If the conductor of K is $m \cdot \infty$, the prime in S, which splits completely in $K/\mathbb{Q}$, does not necessarily split completely in $\mathbb{Q}(\zeta_m)$. For example, take $K = \mathbb{Q}(\sqrt{-7})$ and $S = \{\infty, 7, 11\}$. The prime 11 splits completely in $\mathbb{Q}(\sqrt{-7})$, but not in $\mathbb{Q}(\zeta_7)$; therefore, we cannot apply Theorem 3.25.

In other words, in order to deduce the abelian rank one Stark conjecture for all abelian extensions of $\mathbb{Q}$ from Theorem 3.25, one has to show $St(\mathbb{Q}_{\mathfrak{m}}^{\{p\}}/k, S, p)$ where $\mathfrak{m}$ runs over all moduli of the form $m \cdot \infty$ of $\mathbb{Q}$ and p over all finite prime p such that $p \not| \mathfrak{m}_0$. We remind the reader that $\mathbb{Q}_{\mathfrak{m}}^{\{p\}}$ stands for the maximal subfield of $\mathbb{Q}(\zeta_m)$ in which p splits completely. In other words, it is $\mathbb{Q}(\zeta_m)^{G_p}$. Instead of showing this particular result, we will give an explicit Brumer-Stark element for any abelian extension of $\mathbb{Q}$. In particular, this will provide an explicit formula for a Stark unit for any abelian extensions of $\mathbb{Q}$.

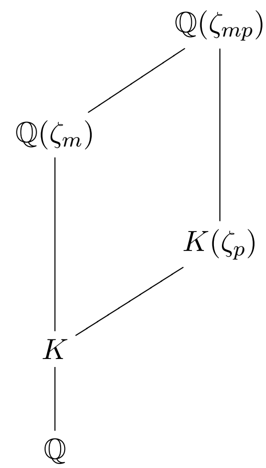
If $K/\mathbb{Q}$ is a finite abelian extension with conductor $m \cdot \infty$, then $S(K/\mathbb{Q}) = S(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. Set $R = S(K/\mathbb{Q})$. First of all, following Weil, we extend the function τ of Section 3.1 multiplicatively to any fractional ideal relatively prime to m. Consider the inclusion of ideals $\iota : I_K \longrightarrow I_{\mathbb{Q}(\zeta_m)}$. If $\mathfrak{p}$ is a prime ideal of K coprime to m (that is unramified), define

$$\tau_K(\mathfrak{p}) = \tau(\iota(\mathfrak{p})) = \prod_{\mathfrak{P} \mid \mathfrak{p}} \tau(\mathfrak{P}),$$

and

$$\varepsilon_K(\mathfrak{p}) = \tau_K(\mathfrak{p})^{w_K}.$$

Consider the following diagram of fields



We remark that $\mathfrak{p}$ is unramified in $\mathbb{Q}(\zeta_m)/K$, but totally ramified in $K(\zeta_p)$; thus, we have $\mathbb{Q}(\zeta_m) \cap K(\zeta_p) = K$.

Proposition 3.26. *With the notation as above, one has*

(1) $\tau_K(\mathfrak{p}) \in K(\zeta_p)$,

(2) $\varepsilon_K(\mathfrak{p}) \in \mathbb{Q}(\zeta_m)$.

Hence, $\varepsilon_K(\mathfrak{p}) \in K$.

Proof:

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{mp})/K(\zeta_p))$, we have to show that $\tau_K(\mathfrak{p})^\sigma = \tau_K(\mathfrak{p})$. Since $p^* \in K(\zeta_p)$, we just have to show that

$$\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P})\right)^\sigma = \prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P}),$$

but

$$\begin{aligned}\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P})\right)^\sigma &= \prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P})^\sigma \\ &= \prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P}^\sigma) \\ &= \prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P}),\end{aligned}$$

since the restriction map induces and isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_{mp})/K(\zeta_p)) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_m)/K)$ and thus σ induces a permutation of the primes of $\mathbb{Q}(\zeta_m)$ lying above $\mathfrak{p}$. We conclude indeed that $\tau_K(\mathfrak{p}) \in K(\zeta_p)$.

As for the second claim, it is sufficient to show that

$$\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P})\right)^{w_K} \in \mathbb{Q}(\zeta_m).$$

Let $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_{mp})/\mathbb{Q}(\zeta_m))$. Then $t \equiv 1 \pmod{m}$, and Theorem 2.4 implies

$$\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P})\right)^{\sigma_t} = \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}}\right)^{-1}_{\mathbb{Q}(\zeta_m),m} \cdot \prod_{\mathfrak{P}\,|\,\mathfrak{p}} g_m(\mathfrak{P}).$$

Hence, if we show that

$$\prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}}\right)_{\mathbb{Q}(\zeta_m),m}$$

is a root of unity in K, we would be done. By definition, this last algebraic number is a root of unity in $\mathbb{Q}(\zeta_m)$; thus, we just have to show it lies in K. Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/K)$, we have

$$\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}}\right)_{\mathbb{Q}(\zeta_m),m}\right)^{\sigma} = \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}}\right)^{\sigma}_{\mathbb{Q}(\zeta_m),m} = \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t^{\sigma}}{\mathfrak{P}^{\sigma}}\right)_{\mathbb{Q}(\zeta_m),m}.$$

Further

$$\begin{aligned}\prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t^{\sigma}}{\mathfrak{P}^{\sigma}}\right)_{\mathbb{Q}(\zeta_m),m} &= \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}^{\sigma}}\right)_{\mathbb{Q}(\zeta_m),m} \\ &= \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \left(\frac{t}{\mathfrak{P}}\right)_{\mathbb{Q}(\zeta_m),m},\end{aligned}$$

since σ induces a permutation of the primes lying above $\mathfrak{p}$. Thus, $\varepsilon_K(\mathfrak{p}) \in \mathbb{Q}(\zeta_m)$. Combined with the first part of the proposition, we get

$$\varepsilon_K(\mathfrak{p}) \in K.$$

Q.E.D.

By the last proposition, we know that $\varepsilon_K(\mathfrak{p}) \in K$. In fact, $\varepsilon_K(\mathfrak{p})$ is a Brumer-Stark element at $\mathfrak{p}$ as the following proposition shows.

Proposition 3.27. *With the notation as above, we have*

$$\mathfrak{p}^{w_K \theta_{K/\mathbb{Q},R}(0)} = (\varepsilon_K(\mathfrak{p})),$$

whenever $\mathfrak{p}$ *is a prime of* K *relatively prime to* m*.*

Proof:

On one hand

$$\varepsilon_K(\mathfrak{p})^{\frac{w_{\mathbb{Q}(\zeta_m)}}{w_K}} = \prod_{\mathfrak{P}\,|\,\mathfrak{p}} \tau(\mathfrak{P})^{w_{\mathbb{Q}(\zeta_m)}}.$$

On the other hand

$$\begin{aligned}\left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} \tau(\mathfrak{P})^{w_{\mathbb{Q}(\zeta_m)}}\right) &= \left(\prod_{\mathfrak{P}\,|\,\mathfrak{p}} \mathfrak{P}\right)^{w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},R}(0)} \\ &= \left(\mathfrak{p}^{w_K \theta_{K/\mathbb{Q},R}(0)} \cdot O_{\mathbb{Q}(\zeta_m)}\right)^{\frac{w_{\mathbb{Q}(\zeta_m)}}{w_K}}.\end{aligned}$$

Thus, we get the following equality of ideals in K:

$$(\varepsilon_K(\mathfrak{p}))^{\frac{w_{\mathbb{Q}(\zeta_m)}}{w_K}} = \left(\mathfrak{p}^{w_K \theta_{K/\mathbb{Q},R}(0)}\right)^{\frac{w_{\mathbb{Q}(\zeta_m)}}{w_K}},$$

from which we conclude that

$$(\varepsilon_K(\mathfrak{p})) = \mathfrak{p}^{w_K \theta_{K/\mathbb{Q},R}(0)},$$

and this is what we wanted to show (The abelian condition is clear as well as the fact that $\varepsilon_K(\mathfrak{p}) \in K^0$).

Q.E.D.

Remark 5. *If* p *is a rational prime which is relatively prime to* m *and* $K = \mathbb{Q}(\zeta_m)^{G_p}$*, then a Stark unit for* $St(K/\mathbb{Q}, R \cup \{p\}, p, \mathfrak{p})$ *is given by*

$$\varepsilon_K(\mathfrak{p}) = \tau(\mathfrak{P})^{w_K},$$

where $\mathfrak{P}$ *is the unique prime of* $\mathbb{Q}(\zeta_m)$ *lying above* $\mathfrak{p}$*.*

With this last result, we conclude that the Brumer-Stark and the abelian rank one Stark conjectures are true for any abelian extension of $\mathbb{Q}$. The Stark units turn out to be really interesting arithmetic objects coming from cyclotomic units or normalized Gauss sums. The abelian rank one Stark conjecture is only fully known for one other kind of base field: The quadratic imaginary number fields. We refer to [53] for an exposition of the ideas involved. In this case, the prime which splits completely is always the infinite one and the Stark units come from elliptic units. The next natural step is to verify the conjecture when the base field is a real quadratic number field and only one infinite prime becomes a complex prime in the abelian extension. This is still an open problem (and the original one investigated numerically by Stark).

3.7 A reciprocity law satisfied by the Stark unit when the base field is $\mathbb{Q}$

In this section, we deduce a reciprocity law satisfied by the Brumer-Stark element from the reciprocity law presented in Proposition 2.18. This was first pointed out by Hayes in [28]. See also [27]. In this section, it is important to work with the special Brumer-Stark element constructed in Section 3.6. Any other Stark unit differ by a root of unity.

Proposition 3.28. *Let m be a positive integer satisfying $m \not\equiv 2 \pmod 4$, $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(\zeta_m)$ coprime to $w_{\mathbb{Q}(\zeta_m)}$, and $R = S(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. The Brumer-Stark element*

$$\varepsilon_R(\mathfrak{p}) = \tau(\mathfrak{p})^{w_{\mathbb{Q}(\zeta_m)}}$$

satisfies the following reciprocity law. Let $\mathfrak{q}$ be any prime ideal relatively prime to m and p, where $\mathfrak{p} \,|\, p$. For all $\sigma \in G_m$, we have

$$\left(\frac{\varepsilon_R(\mathfrak{p}^\sigma)}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m), w_{\mathbb{Q}(\zeta_m)}} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m), w_{\mathbb{Q}(\zeta_m)}}^{w_{\mathbb{Q}(\zeta_m)}\zeta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},R}(0,\sigma)}.$$

Proof:

Suppose first that m is even. Then we know (see equation (3.2)) that

$$w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},R}(0) = \sum_{\substack{t=1 \\ (t,m)=1}}^{m-1} \left(\frac{m}{2} - t\right) \sigma_t^{-1}.$$

Given $\sigma_t \in G_m$, we compute

$$\begin{aligned}\left(\frac{\varepsilon_R(\mathfrak{p}^{\sigma_t})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} &= \left(\frac{\Phi_m(\mathfrak{p}^{\sigma_t})}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m} \cdot \left(\frac{p^*}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),m}^{\frac{-fm}{2}} \\ &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{p^*}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),2}^{f},\end{aligned}$$

by Proposition 2.18. By Proposition 2.12, we have

$$\left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{p^*}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),2}^{f} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{p^*}{\mathbb{N}(\mathfrak{q})}\right)^{f},$$

where the symbol on the right is the usual Legendre symbol. Further, the usual quadratic reciprocity law leads to

$$\begin{aligned}\left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{p^*}{\mathbb{N}(\mathfrak{q})}\right)^{f} &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{p}\right)^{f} \\ &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{p^f}\right).\end{aligned}$$

Then we compute

$$\begin{aligned}\left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{p^f}\right) &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),2} \\ &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{\frac{m}{2}},\end{aligned}$$

by Proposition 2.11. At last, we have

$$\begin{aligned}\left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{-t} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),m}^{\frac{m}{2}} &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}}^{\frac{m}{2}-t} \\ &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}}^{w_{\mathbb{Q}(\zeta_m)}\zeta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},R}(0,\sigma_t)}.\end{aligned}$$

If m is odd, the proof is similar and left to the reader.

Q.E.D.

Proposition 3.29. *Let m be a positive integer satisfying $m \not\equiv 2 \pmod 4$, $\mathfrak{p}$ a prime ideal of $\mathbb{Q}(\zeta_m)$ coprime to $w_{\mathbb{Q}(\zeta_m)}$, and S any set of places of $\mathbb{Q}$ containing $S(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. The Brumer-Stark element $\varepsilon_S(\mathfrak{p})$ satisfies the following reciprocity law. Let $\mathfrak{q}$ be any* *prime ideal relatively prime to m and p, where $\mathfrak{p} \,|\, p$. Then*

$$\left(\frac{\varepsilon_S(\mathfrak{p}^\sigma)}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{a_S(\sigma)}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}},$$

for all $\sigma \in G_m$, where we set

$$w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0) = \sum_{\sigma \in G_m} a_S(\sigma)\sigma^{-1} \in \mathbb{Z}[G_m].$$

Proof:

Suppose that $S' = S \cup \{\mathfrak{p}\}$ and that the proposition holds for S. From the identity

$$w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S'}(0) = (1-\sigma_{\mathfrak{p}}^{-1})w_{\mathbb{Q}(\zeta_m)}\theta_{\mathbb{Q}(\zeta_m)/\mathbb{Q},S}(0),$$

we get

$$a_{S'}(\sigma) = a_S(\sigma) - a_S(\sigma\sigma_{\mathfrak{p}}^{-1}).$$

moreover, we have

$$\varepsilon_{S'}(\mathfrak{p}^\sigma) = \varepsilon_S(\mathfrak{p}^\sigma)^{1-\sigma_{\mathfrak{p}}^{-1}}.$$

Therefore

$$\left(\frac{\varepsilon_{S'}(\mathfrak{p}^\sigma)}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} = \left(\frac{\varepsilon_S(\mathfrak{p}^\sigma)^{1-\sigma_{\mathfrak{p}}^{-1}}}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}}.$$

Further,

$$\begin{aligned}
\left(\frac{\varepsilon_S(\mathfrak{p}^\sigma)^{1-\sigma_{\mathfrak{p}}^{-1}}}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} &= \left(\frac{\varepsilon_S(\mathfrak{p}^\sigma)}{\mathfrak{q}}\right)_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} \cdot \left(\frac{\varepsilon_S(\mathfrak{p}^{\sigma\sigma_{\mathfrak{p}}^{-1}})}{\mathfrak{q}}\right)^{-1}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} \\
&= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{a_S(\sigma)}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} \cdot \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{-a_S(\sigma\sigma_{\mathfrak{p}}^{-1})}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} \\
&= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{a_S(\sigma)-a_S(\sigma\sigma_{\mathfrak{p}}^{-1})}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}} \\
&= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)^{a_{S'}(\sigma)}_{\mathbb{Q}(\zeta_m),w_{\mathbb{Q}(\zeta_m)}}.
\end{aligned}$$

Q.E.D.

One can descend this reciprocity law to any abelian extension of $\mathbb{Q}$.

Proposition 3.30. *Let $K/\mathbb{Q}$ be an abelian extension of conductor $m \cdot \infty$ and let S be a finite set of primes of $\mathbb{Q}$ containing $S(K/\mathbb{Q})$. The Brumer-Stark element $\varepsilon_{K,S}(\mathfrak{p})$ satisfies the following reciprocity law. Let $\mathfrak{q}$ be any prime of K relatively prime to m and p, where $\mathfrak{p}\,|\,p$. Then*

$$\left(\frac{\varepsilon_{K,S}(\mathfrak{p}^\sigma)}{\mathfrak{q}}\right)_{K,w_K} = \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{K,w_K}^{w_K \zeta_{K/\mathbb{Q},S}(0,\sigma)},$$

for all $\sigma \in G = \mathrm{Gal}(K/\mathbb{Q})$.

Proof:

See Theorem 3.2 of [27].

Q.E.D.

3.8 Equivariant formulation of the abelian rank one Stark conjecture

There are different equivalent formulations of the abelian rank one Stark conjecture. The one we have been working with so far is essentially the one Stark presented in his original paper [53] with some clarifications and precisions given by Tate in his classical book on the subject [58]. It is quite suitable for explicit computations, but there are other formulations perhaps more suitable for a conceptual approach. The purpose of this chapter is to explain one of them, the (S,T)-version. Before introducing this new set of primes T, we reformulate the usual conjecture in a way that put the emphasis on the fact that all our objects are acted upon by a Galois group. This is usually referred to as being an equivariant theory.

The abelian rank one Stark conjecture can be reformulated in terms of the equivariant L-function. The precise statement is as follows.

Conjecture 3.31. *Let K/k be an abelian extension of number fields with Galois group G, S a finite set of primes satisfying hypotheses* (St)*, and w_0 a place of K lying above v_0. Then there exists a S_K-unit ε of K such that*

$$\theta'_{K/k,S}(0) = -\frac{1}{w_K}\sum_{\sigma\in G} \log|\varepsilon^\sigma|_{w_0}\cdot\sigma^{-1}. \tag{3.8}$$

Moreover, if $|S| \geq 3$*, then* ε *satisfies* $|\varepsilon|_w = 1$ *for all* w *not lying above* v_0*, whereas if* $|S| = 2$*,* ε *satisfies* $|\varepsilon|_{w^\sigma} = |\varepsilon|_w$ *for all* $\sigma \in G$*, where* $w \in S_K$ *is a place not lying above* v_0*. At last, the extension* $K(\varepsilon^{1/w_K})/k$ *is abelian.*

Proposition 3.32. *Conjectures 3.7 and 3.31 are equivalent.*

Proof:

One has

$$w_K \cdot \theta'_{K/k,S}(0) = -\sum_{\sigma \in G} \log|\varepsilon^\sigma|_{w_0} \cdot \sigma^{-1},$$

if and only if

$$w_K \cdot \theta'_{K/k,S}(0) \cdot e_\chi = \left(-\sum_{\sigma \in G} \log|\varepsilon^\sigma|_{w_0} \cdot \sigma^{-1} \right) \cdot e_\chi,$$

for all $\chi \in \widehat{G}$. This last equality happens if and only if

$$w_K \cdot L'_{K/k,S}(0, \overline{\chi}) = -\sum_{\sigma \in G} \chi(\sigma^{-1}) \log|\varepsilon^\sigma|_{w_0},$$

for all $\chi \in \widehat{G}$. The equivalence between the two conjectures should then be clear.

Q.E.D.

We shall study further the right-hand side of equation (3.8). In the semi-simple algebra $\mathbb{C}[G]$, we have the usual orthogonal idempotents

$$e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma^{-1} \in \mathbb{C}[G].$$

We are interested in the L-functions having order of vanishing precisely one at $s = 0$. This motivates the following definition.

Definition 3.33. Given an abelian extension K/k with Galois group G and a finite set S of primes of k containing $S(K/k)$, we define

$$\widehat{G}_{1,S} = \{\chi \in \widehat{G} \,|\, r_S(\chi) = 1\} \text{ and } e_{1,S} = \sum_{\chi \in \widehat{G}_{1,S}} e_\chi.$$

The element $e_{1,S}$ is an idempotent as well.

Lemma 3.34. *We have* $e_{1,S} \in \mathbb{Q}[G]$*.*

Proof:

Let M be a cyclotomic field (inside $\mathbb{C}$) containing $\mathbb{Q}(\chi)$ for all $\chi \in \widehat{G}$. Then $\mathrm{Gal}(M/\mathbb{Q})$ not only does it act on $\widehat{G}$, but also on $\widehat{G}_{1,S}$, the action being given by $\alpha \cdot \chi = \alpha \circ \chi$. This is because given any $\alpha \in \mathrm{Gal}(M/\mathbb{Q})$, one has $\mathrm{Ker}(\chi) = \mathrm{Ker}(\alpha \circ \chi)$ and $r_S(\chi)$ is given by Theorem 3.5 (and also $\chi_1 = \alpha \circ \chi_1$ for all $\alpha \in \mathrm{Gal}(M/\mathbb{Q})$). Now

$$e_{1,S} = \sum_{\chi \in \widehat{G}_{1,S}} e_\chi = \frac{1}{|G|} \sum_{\chi \in \widehat{G}_{1,S}} \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma^{-1} = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma) \cdot \sigma^{-1}.$$

Hence, it suffices to show that

$$\sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma) \in \mathbb{Q},$$

for all $\sigma \in G$. Let $\alpha \in \mathrm{Gal}(M/\mathbb{Q})$, then

$$\alpha \left(\sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma) \right) = \sum_{\chi \in \widehat{G}_{1,S}} \alpha \cdot \chi(\sigma) = \sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma),$$

since α induces a bijection on $\widehat{G}_{1,S}$. We can conclude the desired result.

Q.E.D.

Remark 6. *Since*

$$\sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma),$$

is also an algebraic integer, it lies in $\mathbb{Z}$. *Thus,* $|G| \cdot e_{1,S} \in \mathbb{Z}[G]$.

Given any $\mathbb{Q}[G]$-module M, following [37], we define $M_{1,S} = e_{1,S}M$. If we view M inside of $\mathbb{C}M$, we have

$$M_{1,S} = \{m \in M \mid e_\chi m = 0, \text{ for all } \chi \notin \widehat{G}_{1,S}\}.$$

If M is a $\mathbb{Z}[G]$-module, the map $M \longrightarrow \mathbb{Q}M$ defined by $m \mapsto 1 \otimes m$ will be denoted by $m \mapsto \tilde{m}$. We remark that the kernel of this last map is precisely the $\mathbb{Z}$-torsion in M. We shall identify the $\mathbb{Z}[G]$-module $E_{K,S}/\mu_K$ with $\tilde{E}_{K,S}$ inside $\mathbb{Q}E_{K,S}$ in the obvious way.

Definition 3.35. Let K/k be an abelian extension of number field, S a finite set of primes of k containing the archimedean places, and v_0 a place in S which splits completely in K/k. We also fix a place w_0 of K lying above v_0. We define a $\mathbb{Z}[G]$-module morphism

$$R_{K/k,w_0} : E_{K,S} \longrightarrow \mathbb{C}[G], \text{ by the formula } u \mapsto -\sum_{\sigma \in G} \log |u^\sigma|_{w_0} \cdot \sigma^{-1},$$

which we extend by $\mathbb{C}$-linearity to $\mathbb{C}E_{K,S}$, and we call it the equivariant regulator.

Remark 7. *It is simple to check that $R_{K/k,w_0}$ is in fact a $\mathbb{Z}[G]$-module morphism.*

The equivariant regulator can be expressed in terms of the Dirichlet logarithmic map in the following way. We shall use the same notation as in [58] and denote the free abelian group on the places of S_K by Y_{S_K}. It is a $\mathbb{Z}[G]$-module and we have the usual surjective trace map $s : Y_{S_K} \longrightarrow \mathbb{Z}$, defined by $w \mapsto 1$ for all $w \in S_K$. The kernel of this map is denoted by X_{S_K}. In summary, we have the following short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow X_{S_K} \longrightarrow Y_{S_K} \xrightarrow{s} \mathbb{Z} \longrightarrow 0.$$

The $\mathbb{Z}[G]$-module structure of $E_{K,S}$ is really difficult to understand, but on the other hand the $\mathbb{Z}[G]$-module structure of X_{S_K} is much simpler. Recall that we have the Dirichlet logarithmic map $\lambda_{K/k,S} : E_{K,S} \longrightarrow \mathbb{C}X_{S_K}$ defined by

$$u \mapsto \sum_{w \in S_K} \log |u|_w \cdot w.$$

For all $v \in S$, fix a place $w = w(v) \in S_K$ lying above v, and let

$$l_w(u) = \frac{1}{|G_v|} \sum_{\sigma \in G} \log |u^\sigma|_w \cdot \sigma^{-1}.$$

A simple computation shows that the Dirichlet map can be rewritten as

$$\lambda_{K/k,S}(u) = \sum_{v \in S} l_w(u) \cdot w.$$

It is easily seen to be a $\mathbb{Z}[G]$-module morphism. After extending $\lambda_{K/k,S}$ to $\mathbb{C}E_{K,S}$ by $\mathbb{C}$-linearity, we get an isomorphism of $\mathbb{C}[G]$-modules. This is the content of Dirichlet's theorem on the structure of the group of S-units. If w_0 is any place in S_K, following

Rubin (see [42]), we define a map $\phi_{w_0} : Y_{S_K} \longrightarrow \mathbb{Z}[G]$ on the generators of Y_{S_K} by

$$\phi_{w_0}(w) = -\sum_{\substack{\gamma \in G \\ \gamma \cdot w_0 = w}} \gamma.$$

It is a $\mathbb{Z}[G]$-module morphism and we remark that if w is another place of S_K not lying above the place of k lying below w_0, then $\phi_{w_0}(w) = 0$. On the other hand, if the place lying below w_0 splits completely then $\phi_{w_0}(w_0) = -1$. The extension of ϕ_{w_0} to a $\mathbb{C}[G]$-module morphism from $\mathbb{C}Y_{S_K}$ to $\mathbb{C}[G]$ will be denoted by the same symbol ϕ_{w_0}.

Proposition 3.36. *Let K/k be an abelian extension of number fields with Galois group G and S a finite set of primes of k containing $S(K/k)$. Assume S contains a split prime v_0 and let $w_0 \in S_K$ be a fixed place above v_0. With the notation as above, we have*

$$R_{K/k,w_0} = \phi_{w_0} \circ \lambda_{K/k,S}.$$

Proof:

Indeed,

$$\begin{aligned} \phi_{w_0} \circ \lambda_{K/k,S}(u) &= \phi_{w_0}\left(\sum_{w \in S_K} \log|u|_w \cdot w\right) \\ &= \phi_{w_0}\left(\sum_{v \in S} l_w(u) \cdot w\right) \end{aligned}$$

Further

$$\begin{aligned} \phi_{w_0}\left(\sum_{v \in S} l_w(u) \cdot w\right) &= \sum_{v \in S} l_w(u) \cdot \phi_{w_0}(w) \\ &= -l_{w_0}(u) \\ &= -\sum_{\sigma \in G} \log|u^\sigma|_{w_0} \cdot \sigma^{-1} \\ &= R_{K/k,w_0}(u) \end{aligned}$$

Q.E.D.

Needless to say, being a $\mathbb{C}[G]$-module isomorphism, the logarithm map $\lambda_{K/k,S}$ induces another isomorphism of $\mathbb{C}[G]$-modules

$$\lambda_{K/k,S} : (\mathbb{C}E_{K,S})_{1,S} \xrightarrow{\simeq} (\mathbb{C}X_{S_K})_{1,S}.$$

What about ϕ_{w_0} when restricted to $(\mathbb{C}X_{S_K})_{1,S}$? Does it induce an isomorphism as well?

Proposition 3.37. *Let K/k be an abelian extension of number fields with Galois group G, S a finite set of primes satisfying the hypotheses* (St)*, and w_0 a place of K lying above v_0. The equivariant regulator induces an isomorphism*

$$R_{K/k,w_0} : \left(\mathbb{C}E_{K,S}\right)_{1,S} \xrightarrow{\simeq} \left(\mathbb{C}[G]\right)_{1,S}.$$

Proof:

We use Proposition 3.36. It suffices to show that ϕ_{w_0} induces an isomorphism

$$(\mathbb{C}X_{S_K})_{1,S} \xrightarrow{\simeq} (\mathbb{C}[G])_{1,S}.$$

We remark that the number of L-functions with order of vanishing precisely one is equal to all the following quantities

$$\dim_{\mathbb{C}}(\mathbb{C}E_{K,S})_{1,S} = \dim_{\mathbb{C}}(\mathbb{C}X_{S_K})_{1,S} = \dim_{\mathbb{C}}(\mathbb{C}[G])_{1,S}.$$

This is due to the fact that given $\chi \in \widehat{G}$, we have

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \dim_{\mathbb{C}}(\mathbb{C}E_{K,S} \cdot e_\chi),$$

by Theorem 3.6. Thus by finite dimensional vector space theory, it suffices to show the surjectivity of the map ϕ_{w_0}. This is clear, since for any character $\chi \in \widehat{G}$ and any place $w \in S_K$ not lying above v_0, one has

$$\phi_{w_0}(e_\chi(w - w_0)) = e_\chi \phi_{w_0}(w - w_0) = -e_\chi.$$

Q.E.D.

Lemma 3.38. *Let K/k be an abelian extension of number fields with Galois group G. Let S be a finite set of primes satisfying the hypotheses* (St)*. Then*

$$\theta'_{K/k,S}(0) \in \left(\mathbb{C}[G]\right)_{1,S}.$$

Proof:

This lemma is clear if one notes that $r_S(\chi) = r_S(\overline{\chi})$ for all character χ, since χ and $\overline{\chi}$ have the same kernel.

Q.E.D.

The abelian rank one Stark conjecture can be reformulated more succinctly as follows.

Conjecture 3.39. *Let K/k be an abelian extension of number fields with Galois group G and let S be a finite set of primes satisfying hypotheses* (St)*. We fix a place w_0 of K lying above v_0. Then there exists a unique $u \in (\tilde{E}_{K,S})_{1,S}$ such that*

$$\theta'_{K/k,S}(0) = \frac{1}{w_K} \cdot R_{K/k,w_0}(u).$$

Moreover, if $\varepsilon \in E_{K,S}$ is such that $\tilde{\varepsilon} = u$, then the extension $K(\varepsilon^{1/w_K})/k$ is abelian.

Remark 8. *The uniqueness of a Stark unit up to a root of unity follows now from Proposition 3.37 since we require $u \in (\tilde{E}_{K,S})_{1,S}$. The fact that this is the same as asking $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above v_0 in the case $|S| \geq 3$, and $|\varepsilon|_{w^\sigma} = |\varepsilon|_w$ for all $\sigma \in G$ where w does not lie above v_0 in the case $|S| = 2$, is contained in Proposition 3.40 below. We will keep both ways of dealing with the uniqueness of a Stark unit in mind.*

Proposition 3.40. *Conjectures 3.31 and 3.39 are equivalent.*

Proof:

Suppose first that $|S| \geq 3$ and Conjecture 3.31 is satisfied with Stark unit ε. We just have to show that $\tilde{\varepsilon}$ satisfies

$$e_\chi \cdot \tilde{\varepsilon} = 0,$$

for all $\chi \notin \widehat{G}_{1,S}$. We use the Dirichlet logarithm map $\lambda_{K/k,S}$. Since it induces an isomorphism of $\mathbb{C}[G]$-modules between $\mathbb{C}E_{K,S}$ and $\mathbb{C}X_{S_K}$, we need to show

$$e_\chi \cdot \lambda_{K/k,S}(\varepsilon) = 0,$$

for all $\chi \notin \widehat{G}_{1,S}$. Let $\chi \notin \widehat{G}_{1,S}$ which is the same as saying that the order of vanishing of $L_{K/k,S}(s,\chi)$ is greater than or equal to two. Then

$$\begin{aligned} e_\chi \cdot \lambda_{K/k,S}(\varepsilon) &= e_\chi \sum_{w \in S_K} \log|\varepsilon|_w \cdot w \\ &= e_\chi \sum_{w \mid v_0} \log|\varepsilon|_w \cdot w, \quad \text{since } |\varepsilon|_w = 1 \text{ for all } w \nmid v_0 \\ &= e_\chi \sum_{\sigma \in G} \log|\varepsilon^{\sigma^{-1}}|_{w_0} \sigma \cdot w_0 \\ &= \sum_{\sigma \in G} \chi(\sigma) \log|\varepsilon^{\sigma^{-1}}|_{w_0} \cdot w_0 \cdot e_\chi \\ &= -w_K L'_{K/k,S}(0,\overline{\chi}) \cdot w_0 \cdot e_\chi \\ &= 0, \end{aligned}$$

since $r_S(\chi) = r_S(\overline{\chi})$.

Conversely, still under the assumption $|S| \geq 3$, suppose that Conjecture 3.39 holds true. If $\tilde{\varepsilon} = u$, we just have to show that $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above v_0. We have

$$e_{1,S} \cdot \tilde{\varepsilon} = \tilde{\varepsilon}.$$

By the remark after Lemma 3.34, we know that

$$|G| e_{1,S} \in \mathbb{Z}[G],$$

and thus there exists a root of unity $\zeta \in \mu_K$ such that

$$\varepsilon^{|G| e_{1,S}} = \zeta \cdot \varepsilon^{|G|}.$$

Let $w \in S_k$ be a place lying above $v \neq v_0$. Then

$$\begin{aligned} |\varepsilon|_w^{|G|} &= \prod_{\sigma \in G} \left| \varepsilon^{\sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma)} \right|_{w^\sigma} \\ &= \prod_{\sigma \in G/G_v} \left| \varepsilon^{\sum_{\chi \in \widehat{G}_{1,S}} \chi(\sigma) \sum_{\tau \in G_v} \chi(\tau)} \right|_{w^\sigma}. \end{aligned}$$

Now, since $|S| \geq 3$, the trivial character is not in $\widehat{G}_{1,S}$. Moreover, if $\widehat{G}_{1,S} \neq \varnothing$ and

$\chi \in \widehat{G}_{1,S}$, the unique decomposition group contained in $\mathrm{Ker}(\chi)$ is G_{v_0}; thus, we have

$$\sum_{\tau \in G_v} \chi(\tau) = 0.$$

We conclude that

$$|\varepsilon|_w^{|G|} = 1,$$

and this is what we wanted to show.

If $|S| = 2$, the proof is left to the reader.

Q.E.D.

3.9 The (S,T)-version of the abelian rank one Stark conjecture

In this section, we introduce another finite set of primes T, and their associated (S,T)-imprimitive L-functions. This will lead us to explain the (S,T)-version of both the abelian rank one Stark conjecture and the Brumer-Stark conjecture.

The usual way to extend the Riemann zeta function to a meromorphic function on the half plane $\mathrm{Re}(s) > 0$ is to consider the Dirichlet series

$$\zeta_{\{2\}}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

which by a standard criteria defines a holomorphic function on the half-plane $\mathrm{Re}(s) > 0$. A simple computation shows that the Riemann zeta function satisfies

$$\zeta_{\{2\}}(s) = (1 - 2^{1-s}) \cdot \zeta(s),$$

and thus we get a meromorphic extension of ζ to this half-plane. The possible poles of ζ are where the zeros of $(1 - 2^{1-s})$ are, namely at

$$s = 1 + \frac{2\pi i k}{\log 2},$$

where $k \in \mathbb{Z}$. In order to rule out all possibilities other than $s = 1$, we consider another function

$$\begin{aligned}\zeta_{\{3\}}(s) &= (1-3^{1-s})\cdot\zeta(s)\\ &= 1+\frac{1}{2^s}-\frac{2}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}-\frac{2}{6^s}+\dots,\end{aligned}$$

which also defines a holomorphic function on the half-plane $\mathrm{Re}(s) > 0$ by the same token. We then conclude that the only possible pole for ζ is at $s = 1$. As it is well known, ζ has indeed a simple pole at $s = 1$, and this can be seen from the fact that $\zeta_{\{2\}}(1) \neq 0$ and $(1-2^{1-s})$ has a simple zero at $s = 1$. These special Euler factors of the shape

$$(1-p^{1-s}),$$

where p is a prime number, will play a special role in the sequel.

Definition 3.41. Let K be a number field, S a finite set of primes of K containing the archimedean places, and T another finite set of finite primes of K satisfying $S \cap T = \varnothing$. The (S,T)-*imprimitive Dedekind zeta function* is defined as

$$\begin{aligned}\zeta_{K,S,T}(s) &= \prod_{\mathfrak{p}\in T}(1-\mathbb{N}(\mathfrak{p})^{1-s})\cdot\zeta_{K,S}(s)\\ &= \prod_{\mathfrak{p}\in T}(1-\mathbb{N}(\mathfrak{p})^{1-s})\cdot\prod_{p\in S}\left(1-\frac{1}{\mathbb{N}(\mathfrak{p})^s}\right)\cdot\zeta_K(s).\end{aligned}$$

It $T \neq \varnothing$, then $\zeta_{K,S,T}$ is now a holomorphic function on the whole complex plane since the simple pole of $\zeta_{K,S}$ is canceled by the zeros at $s = 1$ of the new extra Euler factors associated to primes in T. Therefore, it does not make sense to talk about the residue of $\zeta_{K,S,T}$ at $s = 1$, but it still makes sense to ask what is the first non-vanishing Taylor coefficient of $\zeta_{K,S,T}$ at $s = 0$. As expected, there is a similar formula as in the classical case involving a regulator and a class number.

Theorem 3.42. *The (S,T)-imprimitive Dedekind zeta function has a zero of order $|S|-1$ at $s = 0$ and its first non-vanishing Taylor coefficient is*

$$-\frac{h_{K,S,T}\cdot R_{K,S,T}}{w_{K,T}}$$

In the classical formulation of the abelian rank one Stark conjecture, the number of roots of unity is sometime troublesome. One advantage to work with this (S,T)-theory, rather than the S-theory, is provided by the following proposition.

Proposition 3.43. *Let K be a number field, S a finite set of primes of K containing the archimedean places, and T another finite set of primes disjoint from S. If either one of the following conditions*

(1) *The set T contains a prime $\mathfrak{p}$ such that $(\mathfrak{p}, w_K) = 1$,*

(2) *The set T contains two primes of different residual characteristic,*

holds, then $w_{K,T} = 1$.

Proof:

If T contains a prime $\mathfrak{p}$ such that $(\mathfrak{p}, w_K) = 1$, we just have to show that all $\zeta \in \mu_{K,T}$ are different modulo $\mathfrak{p}$. This would imply in particular that $\zeta \not\equiv 1 \pmod{\mathfrak{p}}$, but this is indeed true because of Lemma 2.7.

If T contains two primes of different residual characteristic, we know (see the proof of Theorem 2.26) that $1 - \zeta$ is either a unit or lies above a single prime. We conclude that $w_{K,T} = 1$ as well.

Q.E.D.

In the case where $E_{K,S,T}$ has no torsion, the first non-vanishing Taylor coefficient at $s = 0$ has the simple form

$$-h_{K,S,T} \cdot R_{K,S,T},$$

where no denominator appears anymore.

Definition 3.44. Let K/k be an abelian extension of number fields with Galois group G. Let S be a finite set of primes of k containing $S(K/k)$ and let T be another finite set of primes of k such that $S \cap T = \varnothing$. If $\chi \in \widehat{G}$, we define the (S,T)-*imprimitive L-function* associated to χ by the formula

$$L_{K/k,S,T}(s,\chi) = \prod_{\mathfrak{p} \in T} (1 - \chi(\sigma_{\mathfrak{p}})\mathbb{N}(\mathfrak{p})^{1-s}) \cdot L_{K/k,S}(s,\chi)$$

Remark 9. *Since the Euler factors associated to primes in T do not vanish at $s = 0$, we see that*

$$\mathrm{ord}_{s=0} L_{K/k,S,T}(s,\chi) = \mathrm{ord}_{s=0} L_{K/k,S}(s,\chi).$$

Theorem 3.45. *Let K/k be a finite abelian extension of number fields. The (S,T)-imprimitive Dedekind zeta function still satisfies a product formula:*

$$\zeta_{K,S_K,T_K}(s) = \prod_{\chi \in \widehat{G}} L_{K/k,S,T}(s,\chi).$$

Proof:

One has to show the equality

$$\left(1 - \mathbb{N}(\mathfrak{p})^{f_{\mathfrak{p}}(1-s)}\right)^{r_{\mathfrak{p}}} = \prod_{\chi \in \widehat{G}} \left(1 - \chi(\sigma_{\mathfrak{p}})\mathbb{N}(\mathfrak{p})^{1-s}\right),$$

for all $\mathfrak{p} \in T$. The proof is similar to the one of Proposition 2.20 and is left to the reader.

Q.E.D.

Definition 3.46. Given an abelian extension of number fields K/k with Galois group G, a finite set S of primes of k containing $S(K/k)$, and a finite set of primes T satisfying $S \cap T = \varnothing$, we define the (S,T)-*equivariant L-function* in the expected way

$$\theta_{K/k,S,T}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S,T}(s,\chi) e_{\overline{\chi}}.$$

Moreover, for $\sigma \in G$, the (S,T)-*partial zeta functions* $\zeta_{K/k,S,T}(s,\sigma)$ are defined by the following group ring equation

$$\theta_{K/k,S,T}(s) = \sum_{\sigma \in G} \zeta_{K/k,S,T}(s,\sigma)\sigma^{-1}.$$

Proposition 3.47. *Let K/k be an abelian extension of number fields and let S and T be two finite sets of primes of k such that S contains $S(K/k)$ and $S \cap T = \varnothing$. Then*

$$\theta_{K/k,S,T}(s) = \delta_T(s) \cdot \theta_{K/k,S}(s),$$

where

$$\delta_T(s) = \prod_{\mathfrak{p} \in T} (1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})^{1-s}).$$

Proof:

If $T = \{\mathfrak{p}\}$, then

$$\begin{aligned}\theta_{K/k,S,T}(s) &= \sum_{\chi \in \widehat{G}} L_{K/k,S,T}(s,\chi) e_{\overline{\chi}} \\ &= \sum_{\chi \in \widehat{G}} \left(1 - \chi(\sigma_{\mathfrak{p}}) \mathbb{N}(\mathfrak{p})^{1-s}\right) L_{K/k,S}(s,\chi) e_{\overline{\chi}} \\ &= \left(1 - \sigma_{\mathfrak{p}}^{-1} \mathbb{N}(\mathfrak{p})^{1-s}\right) \theta_{K/k,S}(s).\end{aligned}$$

The proof for a general T follows by induction.

Q.E.D.

As in the S-version, we have the equivariant regulator $R_{K/k,w_0}$ which induces an isomorphism (the arguments are the same)

$$R_{K/k,w_0} : (\mathbb{C}E_{K,S,T})_{1,S} \xrightarrow{\simeq} (\mathbb{C}[G])_{1,S}\,.$$

Moreover, since the order of vanishing of our new L-functions do not change, we still have

$$\theta'_{K/k,S,T}(0) \in (\mathbb{C}[G])_{1,S}\,.$$

We are now in the position to state the (S,T)-formulation of the abelian rank one Stark conjecture.

Conjecture 3.48. *Let K/k be a finite abelian extension of number fields with Galois group G. Let S be a finite set of primes of k satisfying (St) and let T be another finite set of primes satisfying $S \cap T = \varnothing$. Suppose also that $E_{K,S,T}$ has no torsion, that is $\mu_{K,T} = 1$. We fix a place w_0 lying above the split prime v_0. Then there exists $\varepsilon = \varepsilon_{S,T} \in (E_{K,S,T})_{1,S}$ such that*

$$\theta'_{K/k,S,T}(0) = R_{K/k,w_0}(\varepsilon).$$

We make some remarks.

(1) We shall denote this conjecture by $St(K/k,S,T,v_0)$ or $St(K/k,S,T,v_0,w_0)$ if we want to emphasize the choice of w_0.

(2) Since $E_{K,S,T}$ has no torsion, $E_{K,S,T} \simeq \tilde{E}_{K,S,T}$; thus, we identify the two in the sequel.

(3) The uniqueness of the Stark unit here follows from the requirement $\varepsilon \in (\tilde{E}_{K,S,T})_{1,S}$ as in Conjecture 3.39. Remark also that since $\mu_{K,T} = 1$, the Stark unit is well-defined and there is no more ambiguity due to roots of unity. It would also be possible to require that $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above v_0 in the case $|S| \geq 3$, and $|\varepsilon|_{w^\sigma} = |\varepsilon|_w$ for all $\sigma \in G$ where $w \in S_K$ is not lying above v_0 in the case $|S| = 2$.

(4) It is not so evident at first to see the abelian condition, but it is also included in this formulation as explained below. We remark that in the original formulation of the conjecture by Stark in [53], it is not predicted that $K(\varepsilon^{1/w_K})/k$ is an abelian extension, but rather only a central extension. Nevertheless, in the cases for which the conjecture has been verified so far, the abelian condition is always satisfied.

Lemma 3.49 (Coates)**.** *Let K/k be an abelian extension of number field with Galois group G, and let S be any finite set of primes containing $S(K/k)$ and the ones dividing w_K. Then $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated over $\mathbb{Z}$ by the elements of the form*

$$\sigma_{\mathfrak{p}} - \mathbb{N}(\mathfrak{p}),$$

where $\mathfrak{p}$ runs over all primes not in S.

Proof:

We show first that if $\mathfrak{p} \notin S$, then $\sigma_{\mathfrak{p}} - \mathbb{N}(\mathfrak{p}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. By definition of the Frobenius automorphism, given $\zeta \in \mu_K$, we have

$$\zeta^{\sigma_{\mathfrak{p}}} \equiv \zeta^{\mathbb{N}(\mathfrak{p})} \pmod{\mathfrak{P}},$$

for all $\mathfrak{P} \mid \mathfrak{p}$, and thus

$$\zeta^{\sigma_{\mathfrak{p}} - \mathbb{N}(\mathfrak{p})} \equiv 1 \pmod{\mathfrak{P}},$$

for all such $\mathfrak{P}$. Now, Lemma 2.7 implies that the roots of unity modulo $\mathfrak{P}$ are distinct and thus we conclude that

$$\zeta^{\sigma_{\mathfrak{p}} - \mathbb{N}(\mathfrak{p})} = 1,$$

which is the same as saying that $\sigma_{\mathfrak{p}} - \mathbb{N}(\mathfrak{p}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$.

Now, given $\alpha \in \mathbb{Z}[G]$, write

$$\alpha = \sum_{\sigma \in G} n_\sigma \cdot \sigma.$$

By Cebotarev density theorem, for each $\sigma \in G$, we can fix a prime $\mathfrak{p} \notin S$ such that $\sigma = \sigma_\mathfrak{p}$ and thus we have

$$\alpha = \sum_{\sigma \in G} n_\sigma \cdot \sigma_\mathfrak{p} = n + \sum_{\sigma \in G} n_\sigma (\sigma_\mathfrak{p} - \mathbb{N}(\mathfrak{p})),$$

for some integer n. Since α and the $\sigma_\mathfrak{p} - \mathbb{N}(\mathfrak{p})$ are in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we conclude that $n \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and thus $w_K \,|\, n$.

We shall now show that w_K can be written as a $\mathbb{Z}$-linear combination of elements of the form $1 - \mathbb{N}(\mathfrak{p})$ where $\mathfrak{p} \notin S$ and $\sigma_\mathfrak{p} = 1$, and this will clearly end the proof. In fact, we claim that w_K is the greatest common divisors of the integers $1 - \mathbb{N}(\mathfrak{p})$ where $\mathfrak{p}$ runs over all primes $\mathfrak{p} \notin S$ such that $\sigma_\mathfrak{p} = 1$. Indeed, we have seen previously that if $\sigma_\mathfrak{p} = 1$ then $1 - \mathbb{N}(\mathfrak{p}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and thus we conclude that $w_K \,|\, (1 - \mathbb{N}(\mathfrak{p}))$ for all such $\mathfrak{p}$. Suppose that $d \,|\, (1 - \mathbb{N}(\mathfrak{p}))$ for all $\mathfrak{p} \notin S$ satisfying $\sigma_\mathfrak{p} = 1$. We want to show that $d \,|\, w_K$. Let ζ be any primitive d-root of unity in any extension of K and consider the extension $K(\zeta)/k$. We remark that it is an abelian extension because it is the compositum of two abelian extensions. If we show that $\zeta \in K$ then we would be done. Let thus $\sigma \in \mathrm{Gal}(K(\zeta)/K)$. Then by Cebotarev density theorem again, there exists $\mathfrak{p} \notin S$ such that $(\mathfrak{p}, w_{K(\zeta)}) = 1$ and

$$\sigma = \left(\frac{K(\zeta)/k}{\mathfrak{p}} \right).$$

Since

$$\left(\frac{K(\zeta)/k}{\mathfrak{p}} \right) \Big|_K = \left(\frac{K/k}{\mathfrak{p}} \right),$$

and σ is the identity on K, we conclude that

$$\left(\frac{K/k}{\mathfrak{p}} \right) = 1,$$

and thus $d \,|\, (1 - \mathbb{N}(\mathfrak{p}))$. This in turns implies that σ acts trivially on ζ. Indeed, by definition of the Frobenius automorphism, we have

$$\left(\frac{K(\zeta)/k}{\mathfrak{p}}\right) \cdot \zeta \equiv \zeta^{\mathbb{N}(\mathfrak{p})} \equiv \zeta^{1+ds} \equiv \zeta \pmod{\mathfrak{P}},$$

for all prime ideals $\mathfrak{P}$ of $K(\zeta)$ lying above $\mathfrak{p}$ and for some integer s. Again, by Lemma 2.7, we conclude that

$$\left(\frac{K(\zeta)/k}{\mathfrak{p}}\right) \cdot \zeta = \zeta,$$

and we thus have that $K(\zeta) = K$. This concludes the proof.

Q.E.D.

This last lemma is usually attributed to Coates (see [9]).

Remark 10. *Suppose we have a finite abelian extension of number fields K/k and let L be such that $k \subseteq L \subseteq K$. Let $G = \mathrm{Gal}(K/k)$ and $\Gamma = \mathrm{Gal}(L/k)$, then the restriction map*

$$\pi : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[\Gamma],$$

maps $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ onto $\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$. This is clear from the last lemma using the usual restriction property of the Frobenius automorphism.

Lemma 3.49 clearly implies the following lemma.

Lemma 3.50. *Let K/k be an abelian extension of number fields and let S be a finite set of primes of k containing $S(K/k)$. Then $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated over $\mathbb{Z}[G]$ by the elements of the form*

$$\delta_T := \delta_T(0) = \prod_{\mathfrak{p} \in T} (1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

where T runs over all finite sets of primes in k satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$.

Proposition 3.51. *$St(K/k, S, v_0)$ is equivalent to $St(K/k, S, T, v_0)$, where T runs over all finite set of primes satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$.*

Proof:

One could consult [39] or read Section 5.3 below.

Q.E.D.

We also state the (S, T)-version of the Brumer-Stark conjecture.

Conjecture 3.52. *Let K/k be a finite abelian extension of number fields, R a finite set of primes containing $S(K/k)$ and let T be another finite set of primes satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. Suppose also that $|R| \geq 2$. Given $\mathfrak{a} \in I_{K,\mathfrak{m}(T_K)}$, there exists $\varepsilon_{R,T}(\mathfrak{a}) \in K^\times$ such that*

(1) $\theta_{K/k,R,T}(0) \in \mathbb{Z}[G]$,

(2) $\mathfrak{a}^{\theta_{K/k,R,T}(0)} = (\varepsilon_{R,T}(\mathfrak{a}))$,

(3) $\varepsilon_{R,T}(\mathfrak{a}) \in K^0$,

(4) $\varepsilon_{R,T}(\mathfrak{a}) \equiv 1 \mod {}^\times\mathfrak{P}$, *for all* $\mathfrak{P} \in T_K$.

Point (1) is true due to Theorem 3.1. Remark also that this conjecture predicts in particular that $\theta_{K/k,R,T}(0)$ annihilates $Cl_{K,T}$.

3.10 The Gross conjecture

In this section we state a conjecture formulated by Gross in [24] which gives extra information about the Stark unit in the (S,T)-version of the abelian rank one Stark conjecture. In this section, we suppose that $|S| \geq 3$.

Let K/k be an abelian extension of number fields and suppose that L/k is another abelian extension satisfying $k \subseteq L \subseteq K$. Let v be a place of k which splits completely in L and fix a place $w \in S_L$ lying above v. Let S be a finite set of places of k containing $S(K/k)$, and let T be a finite set of finite places satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. We also let

$$rec_w : L_w^\times \longrightarrow \mathrm{Gal}(K/L),$$

be the local reciprocity map of local class field theory which we view as taking values in $\mathrm{Gal}(K/L)$ rather than in the Galois group of the corresponding local extension (in the usual way). Note that since v splits in L/k, one has $L_w = k_v$.

Conjecture 3.53. *With the notation as above, suppose that $St(L/k,S,T,v)$ is true with Stark unit $\varepsilon_{S,T}$. Then this unit satisfies*

$$rec_w(\varepsilon_{S,T}^{\gamma^{-1}}) = \prod_{\substack{\sigma \in \mathrm{Gal}(K/k) \\ \sigma|_L = \gamma}} \sigma^{\zeta_{K/k,S,T}(0,\sigma^{-1})},$$

for all $\gamma \in \mathrm{Gal}(L/k)$.

We shall denote this conjecture by $Gr(K/L/k, S, T, v)$.

3.11 Current state of knowledge

In this section, we list some landmarks regarding the following four conjectures:

(1) The Brumer conjecture.

(2) The Brumer-Stark conjecture.

(3) The abelian rank one Stark conjecture.

(4) The Gross conjecture.

This list is not meant to be exhaustive and we restrict ourselves to the number field case.

1970 Armand Brumer proposes his conjecture. We did not find it in print, but John Coates in [9] refers to a 1970 Ph.D. thesis by D. Rideout at McGill University titled *A generalization of Stickelberger's theorem*.

1979 Pierrette Cassou-Noguès gives a proof in [3] of the integrality and p-adic interpolation properties of special values of partial zeta functions at non-positive integers building on previous works of Takuro Shintani.

1979 The Gross-Koblitz formula linking the first derivative of a p-adic L-function over $\mathbb{Q}$ to the p-adic logarithm of a Gauss sum is proved in [25].

1980 The abelian rank one Stark conjecture appeared in print for the first time in [53]. The original formulation contains a central condition rather than an abelian condition. The place which splits completely is an infinite place. The conjecture is shown to be true when the base field is $\mathbb{Q}$ using cyclotomic units and when the base field is a quadratic imaginary number field using the theory of complex multiplication and elliptic units.

1980 In [47], Takuro Shintani links his previous work on partial zeta functions to the abelian rank one Stark conjecture.

1980 Pierre Deligne and Kenneth Ribet give a proof of the integrality and p-adic interpolation properties of special values of partial zeta functions at non-positive integers ([14]). Compare to [3], their work gives some extra 2-adic congruences.

1981 John Tate allows the split prime in the abelian rank one Stark conjecture to be finite and introduces the Brumer-Stark conjecture ([56] and [57]).

1982 Benedict Gross introduces a p-adic refinement of the abelian rank one Stark conjecture in [23] and uses his 1979 work on the Gross-Koblitz formula ([25]) to prove it when the base field is $\mathbb{Q}$.

1984 The classical book by John Tate on the subject appears in print ([58]).

1984 Following a suggestion of John Tate, Jonathan Sands shows that the Brumer-Stark and the abelian rank one Stark conjectures hold for a big family of abelian extensions of exponent two in [43] and [45].

1988 Motivated by the work of Barry Mazur and John Tate on refinements on the Birch and Swinnerton-Dyer conjecture for elliptic curves, Benedict Gross introduces in [24] what we call in this thesis the Gross conjecture.

1990 Andrew Wiles introduces techniques from Iwasawa theory in order to study the Brumer conjecture. He introduces a new idea to circumvent the problem of trivial zeros ([65]).

1996 Karl Rubin states a higher order of vanishing Stark conjecture "over $\mathbb{Z}$" in [42] which generalizes the abelian rank one Stark conjecture.

1998 Following a suggestion of John Tate, and building on the work of Jonathan Sands when the base field is $\mathbb{Q}$ ([43]), David Hayes proves in [29] a base change property for the conjecture of Brumer-Stark. This work was extended by Cristian Popescu to the higher order of vanishing situation in [37].

2000 Building on the work of Andrew Wiles in [65], Cornelius Greither shows the Brumer conjecture for a class of extensions which he calls "nice". This work was extended to the Rubin-Stark conjecture by Cristian Popescu in [38].

2003 The Gross conjecture makes sense when the prime involved is a real infinite place. Using the 2-adic congruences of Pierre Deligne and Kenneth Ribet, Michael Reid proves in [41] the Gross conjecture for CM-extensions when the prime involved is a real infinite place.

2006 Samit Dasgupta and Henri Darmon propose a conjectural construction of p-units in narrow ring class fields of real quadratic number fields in [11] ,using modular symbols. Their conjecture implies the prime-to-2 part of the Brumer-Stark conjecture for these particular extensions. Also, it provides strengthening of the Gross conjecture. Hugo Chapdelaine extended in [4] this work to narrow ray class fields.

2007 David Burns shows in [2] that these four conjectures follow from the very general Equivariant Tamagawa Number Conjecture of Bloch and Kato.

2008 Building on the work of Takuro Shintani [47] and Pierrette Cassou-Noguès [3], Samit Dasgupta proposes a conjectural formula for the Gross-Stark unit for any totally real number field as a base field.

2010 In [22], Cornelius Greither and Cristian Popescu proves an Equivariant Main Conjecture in equivariant Iwasawa theory. They use these results and Iwasawa co-descent to prove refinements of the imprimitive Brumer-Stark Conjecture, away from their 2-primary components under the assumption that the appropriate classical Iwasawa μ-invariants vanish (as conjectured by Iwasawa). The minus part of the Gross conjecture follows as well from their main result. The proof of the Brumer-Stark conjecture in the case of function fields given by Tate and Deligne, and presented in [58], was a motivation for their work.

Chapter 4

The extended abelian rank one Stark conjecture

4.1 Formulation of the extended abelian rank one Stark conjecture

In the classical formulation of the abelian rank one Stark conjecture, one assumes that S contains at least one place which splits completely in the relevant abelian extension. This condition guarantees that all the S-imprimitive L-functions associated to non-trivial characters have order of vanishing greater than or equal to one because of Theorem 3.5. On the other hand, it is easy to find examples of finite sets of primes S containing the ramified ones and the archimedean ones for which all the S-imprimitive L-functions associated to non-trivial characters have order of vanishing at least one, but such that S does not contain any prime which splits completely.

For example, one can take $K = \mathbb{Q}(\sqrt{-3}, \sqrt{5})/\mathbb{Q}$ and $S = \{\infty, 3, 5, 7, 17\}$. Everything we need to know about quadratic number fields is contained in Hilbert's Zahlbericht, [31]. Using the usual arithmetic of quadratic number fields, we find

$$G_3 = G_5 = \mathrm{Gal}(K/\mathbb{Q}),\ G_\infty = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5})),$$

and

$$G_7 = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3})),\ G_{17} = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-15})).$$

Every non-trivial character of this Galois group contains at least one decomposition group; thus, all imprimitive L-functions associated to non-trivial characters have order of vanishing at least one by Theorem 3.5. Here, since $|S| \geq 2$, the order of vanishing of $L_{K/k,S}(s, \chi_1)$ is at least one as well. Nevertheless, no prime of S splits completely in K.

Therefore, it is natural to try to find a formula for $L'_{K/k,S}(0, \chi)$ in this more general case. This is what Erickson did in [19] and what we present in this section.

Definition 4.1. Let K/k be an abelian extension of number fields with Galois group G and let Γ be any subset of $\widehat{G}$. Let S be any finite set of primes of k (perhaps not containing the ramified nor the archimedean primes). We say that S is a 1-*cover* for Γ if the following two conditions hold:

(1) For all non-trivial $\chi \in \Gamma$, there exists at least one prime $v \in S$ such that $G_v \subseteq \mathrm{Ker}(\chi)$,

(2) If the trivial character is in Γ, then $|S| \geq 2$.

Remark 11. *In the case where $S \supseteq S(K/k)$ and $\Gamma = \widehat{G}$, this precisely means that*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s, \chi) \geq 1,$$

for all $\chi \in \widehat{G}$, by Theorem 3.5. The extended abelian rank one Stark conjecture will be a similar statement as the usual abelian rank one Stark conjecture where the set S containing a prime which splits completely will be replaced by a 1-cover of $\widehat{G}$ containing $S(K/k)$.

Definition 4.2. As usual, let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. We define S_{min} as follows: It consists of all primes $v \in S$ for which there exists $\chi \in \widehat{G}_{1,S}$, $\chi \neq \chi_1$, such that $G_v \subseteq \mathrm{Ker}(\chi)$.

Remark 12. *The denotation S_{min} is used because, as it is not hard to see, S_{min} is the minimal 1-cover of $\widehat{G}_{1,S} \smallsetminus \{\chi_1\}$ (see [17] if needed). Note also that S_{min} is not necessarily itself a 1-cover of S.*

As an example, suppose that $S \supseteq S(K/k)$ and that S has exactly one prime, say v, which splits completely. The set S is clearly a 1-cover of $\widehat{G}$ (if $|S| \geq 2$). Moreover, if $\widehat{G}_{1,S} \smallsetminus \{\chi_1\} \neq \varnothing$, then $S_{min} = \{v\}$.

If we come back to the example $K/\mathbb{Q}$ with $K = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$ and the 1-cover $S = \{\infty, 3, 5, 7, 17\}$, we see that $S_{min} = \{\infty, 7, 17\}$, since the corresponding decomposition groups of these three primes are precisely the three subgroups of order two of $\mathrm{Gal}(K/\mathbb{Q})$ and each such subgroup is precisely the kernel of a non-trivial character. In this case, S_{min} is itself a 1-cover of G. On the other hand, if we take $S = \{\infty, 3, 5, 7, 11, 17\}$, then $S_{min} = \{7, 17\}$ which is not a 1-cover of G.

Our first example was a biquadratic extension, and this is because it is the kind of extensions of smallest degree for which there exist 1-covers not containing a split prime. Indeed, we have:

Proposition 4.3. *Suppose that K/k is a finite cyclic extension of number fields with Galois group G. Then any 1-cover S for $\widehat{G}$ containing $S(K/k)$ has to contain at least one prime which splits completely. In the case where it contains exactly one split prime, say v, we have $S_{min} = \{v\}$.*

Proof:

If G is any cyclic group of order greater than or equal to two, there exists a faithful character, say χ, and this character is necessarily non-trivial. Since S is a 1-cover for $\widehat{G}$, we have that there exists $v \in S$ such that

$$G_v \subseteq \mathrm{Ker}(\chi) = 1.$$

This implies that $G_v = 1$ which is the same as saying that v splits completely. If moreover, v is the only prime in S which splits completely, then $\chi \in \widehat{G}_{1,S}$ and $S_{min} = \{v\}$, since $G_v \subseteq \mathrm{Ker}(\psi)$ for all characters $\psi \in \widehat{G}_{1,S} \smallsetminus \{\chi_1\}$.

Q.E.D.

We can now state the extended abelian rank one Stark conjecture as stated in [19].

Conjecture 4.4. *Let K/k be a finite abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ satisfying $|S| \geq 3$, $S \neq S_{min}$, and $S \supseteq S(K/k)$. For each $v \in S_{min}$, fix a place $w = w(v)$ of K lying above v. Then, there exists a S_K-unit $\varepsilon \in E_{K,S}$ such that*

$$L'_{K/k,S}(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left(\prod_{v \in S_{min}} |\varepsilon^{\sigma}|_w^{1/|G_v|} \right). \tag{4.1}$$

Moreover, the S_K-unit is a S_{min}-unit meaning that $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above a $v \in S_{min}$. At last, the extension $K(\varepsilon^{1/w_K})/k$ is abelian.

Let us make some remarks.

(1) If S is a 1-cover and $|S| = 2$, then it can be shown that S contains a place which splits completely, see [17]. Therefore, we can avoid this case, since it is contained in the usual abelian rank one Stark conjecture.

(2) In the case where $S_{min} = \{v\}$ and v is a prime which splits completely, the extended abelian rank one Stark conjecture reduces to the standard abelian rank one Stark conjecture. In particular, this new conjecture does not say anything new in the case of cyclic extensions.

(3) A unit satisfying equation (4.1) is also called a Stark unit as in the classical abelian rank one Stark conjecture.

(4) We shall denote this conjecture by $ESt(K/k, S)$.

(5) This last conjecture can be reinterpreted in terms of partial zeta functions as it was the case for the abelian rank one Stark conjecture. Using Lemma 3.8 we get that formula (4.1) is equivalent to

$$-w_K \zeta'_{k,S}(0, \tau) = \sum_{v \in S_{min}} \frac{1}{|G_v|} \log |\varepsilon^\tau|_w,$$

for all $\tau \in G$.

(6) The uniqueness of ε does not follow from Kronecker's theorem as in the usual abelian rank one Stark conjecture, since one cannot isolate its various absolute values. The uniqueness rather follows from Proposition 4.9 below.

This conjecture satisfies the usual functoriality properties.

Proposition 4.5. *If $S \subseteq S'$ then $ESt(K/k, S)$ implies $ESt(K/k, S')$. If $k \subseteq K' \subseteq K$ then $ESt(K/k, S)$ implies $ESt(K'/k, S)$.*

Proof:

The proofs are similar to the ones for the usual abelian rank one Stark conjecture. The reader can consult [19] for details. See also Section 5.4 for similar arguments.

Q.E.D.

4.2 Equivariant reformulation of the extended abelian rank one Stark conjecture

The extended abelian rank one Stark conjecture can be reformulated in an equivariant way similarly as we did for the standard abelian rank one Stark conjecture.

Conjecture 4.6. *Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover of $\widehat{G}$ satisfying $|S| \geq 3$, $S \neq S_{min}$, and $S \supseteq S(K/k)$. For each place $v \in S_{min}$, fix a place $w = w(v) \in S_K$ lying above v. Then, there exists a S_K-unit $\varepsilon \in E_{K,S}$ such that*

$$\theta'_{K/k,S}(0) = -\frac{1}{w_K} \cdot \sum_{v \in S_{min}} \frac{1}{|G_v|} \sum_{\sigma \in G} \log |\varepsilon^{\sigma}|_w \cdot \sigma^{-1}.$$

Moreover, $|\varepsilon|_w = 1$ for all $w \in S_K$ not lying above a $v \in S_{min}$, and the extension $K(\varepsilon^{1/w_K})/k$ is abelian.

Proposition 4.7. *Conjectures 4.4 and 4.6 are equivalent.*

Proof:

The proof is similar to the one of Proposition 3.32 and is left to the reader.

Q.E.D.

Definition 4.8. Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. For every $v \in S_{min}$, we fix a place $w = w(v) \in S_K$ lying above v. We define a $\mathbb{Z}[G]$-module morphism

$$R_{K/k,S} : E_{K,S} \longrightarrow \mathbb{C}[G], \text{ by the formula } u \mapsto -\sum_{v \in S_{min}} \frac{1}{|G_v|} \sum_{\sigma \in G} \log |u^{\sigma}|_w \cdot \sigma^{-1},$$

which we extend to $\mathbb{C}E_{K,S}$, and we call it the *equivariant regulator.*

Remark 13. *It is simple to check that $R_{K/k,S}$ is in fact a $\mathbb{Z}[G]$-module morphism.*

Proposition 4.9. *Let K/k be a finite abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. The equivariant regulator induces an isomorphism of $\mathbb{C}[G]$-modules*

$$R_{K/k,S} : (\mathbb{C}E_{K,S})_{1,S} \xrightarrow{\simeq} (\mathbb{C}[G])_{1,S}.$$

Proof:

See Proposition 3.2 of [17].

Q.E.D.

The proof of the following lemma is simple and left to the reader.

Lemma 4.10. *Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. Then*

$$\theta'_{K/k,S}(0) \in (\mathbb{C}[G])_{1,S}.$$

Conjecture 4.6 can be reformulated as follows.

Conjecture 4.11. *Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover of $\widehat{G}$ containing $S(K/k)$. Suppose that S satisfies $|S| \geq 3$ and $S \neq S_{min}$. Then, there exists $\varepsilon \in (\tilde{E}_{K,S})_{1,S}$ such that*

$$\theta'_{K/k,S}(0) = \frac{1}{w_K} \cdot R_{K/k,S}(\varepsilon).$$

Moreover, if $u \in E_{K,S}$ is such that $\tilde{u} = \varepsilon$, then the extension $K(u^{1/w_K})/k$ is abelian.

It is convenient to introduce the following definition as in [17].

Definition 4.12. Let K/k be an abelian extension of number fields and S a 1-cover for $\widehat{G}$ containing $S(K/k)$. The *evaluator* $\varepsilon_{K/k,S}$ is defined to be the unique element in $(\mathbb{C}E_{K,S})_{1,S}$ satisfying

$$\theta'_{K/k,S}(0) = \frac{1}{w_K} \cdot R_{K/k,S}(\varepsilon_{K/k,S}).$$

With this definition, the abelian rank one Stark conjecture amounts to showing that

$$\varepsilon_{K/k,S} \in \tilde{E}_{K,S},$$

and the abelian condition.

So far all known cases of this conjecture use the following proposition (which is a reformulation of the discussion contained in Section 5 of [19]).

Proposition 4.13. *Let K/k be a finite abelian extension of number fields and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. Suppose also that $|S| \geq 3$. For each $v \in S_{min}$*

choose $w = w(v) \in S_K$ *and also* $w' = w'(v) \in S_{K^{G_v}}$ *between* v *and* w*. Then, we have*

$$\varepsilon_{K/k,S} = \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S}, \tag{4.2}$$

where $\varepsilon_{K^{G_v}/k,S} \in (\mathbb{C}E_{K^{G_v},S})_{1,S}$ *satisfies*

$$\theta'_{K^{G_v}/k,S}(0) = \frac{1}{w_{K^{G_v}}} \cdot R_{K^{G_v}/k,w'}(\varepsilon_{K^{G_v}/k,S}).$$

Proof:

Given $\chi \in \widehat{G}$, one has the following formula

$$e_\chi \cdot \varepsilon_{K^{G_v}/k,S} = \begin{cases} 0 & \text{if } \chi\,|_{G_v} \neq 1, \\ e_{\widetilde{\chi}} \cdot \varepsilon_{K^{G_v}/k,S} & \text{if } \chi|_{G_v} = 1, \end{cases} \tag{4.3}$$

where $\widetilde{\chi}$ is the induced character on G/G_v. Indeed, let $\sigma_1, \ldots, \sigma_s$ be coset representatives for G/G_v, then we have

$$\begin{aligned} e_\chi \cdot \varepsilon_{K^{G_v}/k,S} &= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{h \in G_v} \chi(\sigma_i h)\sigma_i^{-1} h^{-1} \varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{h \in G_v} \chi(\sigma_i h)\sigma_i^{-1} \varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \chi(\sigma_i)\sigma_i^{-1} \left(\sum_{h \in G_v} \chi(h) \right) \varepsilon_{K^{G_v}/k,S} \end{aligned}$$

Noting that the quantity in parenthesis is $|G_v|$ or 0 depending on whether or not $G_v \subseteq \mathrm{Ker}(\chi)$, we conclude the validity of formula (4.3).

We prove formula (4.2) one character at the time. Since $|S| \geq 3$, both sides are equal to 0 when multiplied by e_{χ_1}. Let $\chi \in \widehat{G}$ be a non-trivial character and suppose first that $\chi \notin \widehat{G}_{1,S}$, then one has

$$e_\chi \cdot \varepsilon_{K/k,S} = 0,$$

since $\varepsilon_{K/k,S} \in (\mathbb{C}E_{K,S})_{1,S}$. On the right-hand side we have

$$e_\chi \cdot \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S} = e_{\widetilde{\chi}} \cdot \sum_{\chi\,|_{G_v}=1} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S}$$

by formula (4.3). But $e_{\widetilde{\chi}} \cdot \varepsilon_{K^{G_v}/k,S} = 0$ as well since the associated L-function also has a zero or order at least two.

Suppose now that $\chi \in \widehat{G}_{1,S}$, then we have on one hand

$$R_{K/k,S}(e_\chi \cdot \varepsilon_{K/k,S}) = w_K \cdot L'_{K/k,S}(0, \bar{\chi}).$$

If we multiply the right-hand side of formula (4.2) with e_χ, we get

$$\frac{w_K}{w_{K^{G_{v_0}}}} \frac{1}{|G_{v_0}|} \cdot e_\chi \cdot \varepsilon_{K^{G_{v_0}}/k,S},$$

where v_0 is the unique place in S such that $G_{v_0} \subseteq \mathrm{Ker}(\chi)$. We apply now $R_{K/k,S}$ to this last expression. For $v \in S_{min}$, $u \in E_{K^{G_v},S}$ and $\sigma_1, \ldots, \sigma_s$ coset representatives of G/G_{v_0}, we have

$$\begin{aligned} e_\chi \frac{1}{|G_v|} \sum_{\sigma \in G} \log |u^\sigma|_w \cdot \sigma^{-1} &= e_\chi \frac{1}{|G_v|} \sum_{i=1}^{s} \sum_{h \in G_{v_0}} \log |u^{\sigma_i h}|_w (\sigma_i h)^{-1} \\ &= e_\chi \frac{1}{|G_v|} \sum_{i=1}^{s} \left(\sum_{h \in G_{v_0}} h \right) \log |u^{\sigma_i h}|_w \sigma_i^{-1}. \end{aligned}$$

Noting that $e_\chi \sum_{h \in G_{v_0}} h = 0$ if $G_{v_0} \not\subseteq \mathrm{Ker}(\chi)$, we conclude that

$$R_{K/k,S}(e_\chi \cdot \varepsilon_{K^{G_{v_0}}/k,S}) = |G_{v_0}| \cdot R_{K^{G_{v_0}}/k,w'}(\varepsilon_{K^{G_{v_0}}/k,S} \cdot e_{\widetilde{\chi}}) = |G_{v_0}| \cdot w_{K^{G_{v_0}}} \theta'_{K^{G_{v_0}}/k,S}(0) \cdot e_{\widetilde{\chi}}.$$

Therefore

$$R_{K/k,S} \left(\frac{w_K}{w_{K^{G_{v_0}}}} \frac{1}{|G_{v_0}|} \cdot e_\chi \cdot \varepsilon_{K^{G_{v_0}}/k,S} \right) = w_K \cdot \theta'_{K^{G_v}/k,S}(0) \cdot e_{\widetilde{\chi}},$$

and we conclude the equality of the two sides by the inflation property of Artin L-functions.

Q.E.D.

As explained in [19], this proposition can be used as follows. Suppose that the usual abelian rank one Stark conjecture is known for subextensions of K/k. Let $\varepsilon_v \in E_{K^{G_v},S}$ be a Stark unit for $St(K^{G_v}/k, S, v, w')$ that is

$$\tilde{\varepsilon}_v = \varepsilon_{K^{G_v}/k,S}.$$

Suppose we can show

(1) There exists $\eta_v \in K^\times$ such that $\eta_v^{|G_v|} = \varepsilon_v$.

(2) The extension $K(\eta_v^{\frac{1}{w_v}})/k$ is abelian, where we abbreviate $w_v = w_{K^{G_v}}$.

Then we would have

$$\varepsilon_{K/k,S} = \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \tilde{\eta}_v \in \tilde{E}_{K,S},$$

and if $\varepsilon \in E_{K,S}$ is such that $\tilde{\varepsilon} = \varepsilon_{K/k,S}$, the extension $K(\varepsilon^{\frac{1}{w_K}})/k$ would be abelian, since it would be contained in the compositum of the fields $K(\eta_v^{\frac{1}{w_v}})$ as v runs over all places in S_{min}.

At times, this method allows one to prove the conjecture if the usual abelian rank one Stark conjecture is already known for intermediate extensions of K/k.

4.3 The (S,T)-version of the extended abelian rank one Stark conjecture

We can now state the (S,T)-version of the extended abelian rank one Stark conjecture.

Definition 4.14. Let K/k be an abelian extension of number fields and S a 1-cover for $\widehat{G}$ containing $S(K/k)$. Let T be a finite set of primes of k satisfying $S \cap T = \varnothing$. The (S,T)-*evaluator* $\varepsilon_{K/k,S,T}$ is defined to be the unique element in $(\mathbb{C}E_{K,S,T})_{1,S}$ satisfying

$$\theta'_{K/k,S,T}(0) = R_{K/k,S}(\varepsilon_{K/k,S,T}).$$

Conjecture 4.15. *Let K/k be an abelian extension of number fields and S a 1-cover for $\widehat{G}$ containing $S(K/k)$. Suppose that $|S| \geq 3$ and $S \neq S_{min}$. Also, let T be another finite set of primes of k satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. Then*

$$\varepsilon_{K/k,S,T} \in \tilde{E}_{K,S,T}.$$

The counterpart of Proposition 4.13 is given by

Proposition 4.16. *Let K/k be a finite abelian extension of number fields and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. Suppose also that $|S| \geq 3$. Let T be another finite set of primes satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. For each $v \in S_{min}$ choose $w = w(v) \in S_K$ and $w'(v) \in S_{K^{G_v}}$ between v and w. Then*

$$\varepsilon_{K/k,S,T} = \sum_{v \in S_{min}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S,T},$$

where $\varepsilon_{K^{G_v}/k,S,T} \in (\mathbb{C}E_{K^{G_v},S,T})_{1,S}$ satisfies

$$\theta'_{K^{G_v}/k,S,T}(0) = R_{K^{G_v}/k,w'}(\varepsilon_{K^{G_v}/k,S,T}).$$

Proof:

The proof is similar to the one given for Proposition 4.13 and left to the reader. It is contained in [17].

Q.E.D.

4.4 Minimal cocyclic subgroups and 1-covers

There is a notion which is useful in order to understand 1-covers: It is the notion of minimal cocyclic subgroups. Among other things, we use this notion to provide the numerical examples contained in Appendix B.

Definition 4.17. Let G be a finite abelian group. A subgroup H is called *cocyclic* if G/H is a cyclic group. Moreover, if H is cocyclic and if $K \subsetneq H$ implies that G/K is not cyclic, then we say that H is a *minimal cocyclic subgroup.*

Remark 14. *This notion probably exists in group theory, but we did not find it in the literature. The terminology cocyclic has been suggested by my friend Joel Dodge.*

This notion is relevant because of the following reason. Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover of $\widehat{G}$. Let χ be a non-trivial character of G and suppose it does not have order 2. We clearly have

$$\mathrm{Ker}(\chi) \subseteq \mathrm{Ker}(\chi^2),$$

and thus

$$\operatorname{ord}_{s=0}(L_{K/k,S}(s,\chi)) \leq \operatorname{ord}_{s=0}(L_{K/k,S}(s,\chi^2)).$$

Lemma 4.18. *The cocyclic subgroups are precisely the kernels of the characters of G.*

Proof:

Let $H = \operatorname{Ker}(\chi)$ be the kernel of some character χ. Then χ can be viewed as a faithful character of G/H and since the only finite subgroups of $\mathbb{C}^\times$ are cyclic, we conclude that G/H is a cyclic group.

Conversely, if H is a cocyclic subgroup of G, then G/H is a finite cyclic group, and for any such group, there exists a faithful character, say χ. Now, its inflation to G will have kernel precisely equal to H.

Q.E.D.

Theorem 4.19. *Let K/k be an abelian extension of number fields with Galois group G. Let $H_1, \ldots, H_m$ be the minimal cocyclic subgroups of G. A finite set of primes S is a 1-cover of $\widehat{G}$ if and only if for all $i = 1, \ldots, m$ we have*

$$G_v \subseteq H_i,$$

for some $v \in S$ (depending on i) and also $|S| \geq 2$.

Proof:

This follows immediately from Definition 4.1, Lemma 4.18, and the fact that any cocyclic subgroup contains a minimal cocyclic subgroup.

Q.E.D.

Theorem 4.20. *Let K/k be a finite abelian extension of number fields with Galois group $G \neq 1$. Let $H_1, \ldots, H_m$ be the minimal cocyclic subgroups of G and let S be a 1-cover of $\widehat{G}$, then S_{min} consists precisely of the primes $v \in S$ such that there exists a H_i satisfying:*

(1) *$G_v \subseteq H_i$,*

(2) *$G_{v'} \nsubseteq H_i$, for all $v' \in S$, $v' \neq v$.*

Proof:

Let $\chi \in \widehat{G}_{1,S} \smallsetminus \{\chi_1\}$ and let v be the unique place in S satisfying $G_v \subseteq \mathrm{Ker}(\chi)$. First, we want to show that there exists a minimal cocyclic subgroup H such that $G_v \subseteq H$. Let H be a minimal cocyclic subgroup contained in $\mathrm{Ker}(\chi)$. Since S is a 1-cover, there exists, by Theorem 4.19, $v' \in S$ such that $G_{v'} \subseteq H$. If $v' \neq v$, then we would have

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) \geq 2,$$

and this would be a contradiction. Therefore, $v' = v$ and $G_v \subseteq H$. If there were another $v' \in S$ such that $G_{v'} \subseteq H$, it would be again a contradiction with the fact that $\chi \in \widehat{G}_{1,S}$.

Conversely, let $v \in S$ and suppose that there exists a minimal cocyclic subgroup H such that $G_v \subseteq H$ and $G_{v'} \nsubseteq H$ for all $v' \in S$, $v' \neq v$. We want to show that $v \in S_{min}$. By Lemma 4.18, we know that there exists a character χ such that $H = \mathrm{Ker}(\chi)$. If $\chi = \chi_1$, then G itself would be a minimal cocyclic subgroup, and this would imply that $G = 1$ which is excluded. Hence, $\chi \neq \chi_1$. If we show that $\chi \in \widehat{G}_{1,S}$, then we would be done. But this is precisely what the second condition means.

Q.E.D.

The notion of cocyclic subgroup is useful in order to compute explicit examples of 1-covers for a given abelian extension of number fields with Galois group G. One starts by finding the minimal cocyclic subgroups of G. Then one checks whether or not the decomposition groups associated to ramified and archimedean primes are contained in some of these minimal cocyclic subgroups which leads to the following definition.

Definition 4.21. Let K/k be a finite abelian extension of number fields with Galois group G and let S be a 1-cover for G. The minimal cocyclic subgroups containing a decomposition group G_v for some finite ramified or infinite place v in S will be called *distinguished*. If we want to emphasize the G_v contained in the minimal cocyclic subgroup H, we shall say that H is *v-distinguished*. Moreover, the collection of v-distinguished minimal cocyclic subgroups for finite ramified places v (resp. infinite places v) will be referred to as the *finite distinguished* minimal cocyclic subgroups (resp. *infinite distinguished*).

Finally, one fills in S by adding finite unramified primes whose Frobenius automorphisms are contained in the non-distinguished minimal cocyclic subgroups.

Since an example is worth a thousand words, let us look again at the biquadratic extension $K = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$. We denote its Galois group over $\mathbb{Q}$ by G. The cocyclic subgroups are

$$\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3})), \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5})), \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-15})), \text{ and } G.$$

The minimal cocyclic subgroups are the three subgroups of G of order two. There are two ramified primes which are 3 and 5 and their decomposition groups are

$$G_3 = G_5 = G.$$

Moreover, we have

$$G_\infty = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5})),$$

and thus this last subgroup of order two is a distinguished minimal cocyclic subgroup. It is the only one. According to our procedure, we have to find finite unramified primes whose Frobenius automorphisms are contained in the non-distinguished minimal cocyclic subgroups. We can choose 7 and 17 for instance, and this is why $S = \{\infty, 3, 5, 7, 17\}$ is a 1-cover for G.

Definition 4.22. Given a finite abelian group G, we let $n_d(G)$ (or n_d when G is fixed) denotes the number of cyclic subgroups of G of order d. Moreover, we let n_G denotes the number of cyclic subgroups of G. Hence

$$n_G = \sum_{d \mid n} n_d(G).$$

Let n'_d (resp. n'_G) be the number of minimal cocyclic subgroups of order d (resp. of G).

Proposition 4.23. *Let G be a finite abelian group. The number of cocyclic subgroups of cardinality d is equal to $n_{|G|/d}(G)$.*

Proof:

We make use of the perfect pairing coming from duality theory for finite abelian groups:

$$\widehat{G} \times G \longrightarrow \mathbb{C}^\times.$$

The finite cocyclic subgroups of cardinality d of G correspond to finite cyclic subgroups of cardinality $|G|/d$ of $\widehat{G}$. Indeed, if H is a finite cocyclic subgroup of cardinality d, then we have

$$H^{\perp} \simeq \widehat{G/H} \simeq G/H,$$

and thus we conclude that $H^{\perp}$ is cyclic of cardinality $|G|/d$. Conversely, if C is a cyclic subgroup of cardinality $|G|/d$ of $\widehat{G}$, then let χ be a generator of C. We claim that $C^{\perp} = \mathrm{Ker}(\chi)$. The inclusion $C^{\perp} \subseteq \mathrm{Ker}(\chi)$ is obvious from the definition. If $g \in \mathrm{Ker}(\chi)$, then let $\psi \in C$. There exists $i \in \mathbb{Z}$ such that $\psi = \chi^i$ and thus $\psi(g) = 1$ as well, and we conclude that $C^{\perp} = \mathrm{Ker}(\chi)$. Therefore, $C^{\perp}$ is cocyclic of cardinality d.

Q.E.D.

Theorem 4.24. *We have the following formula for any finite abelian group G:*

$$|G| = \sum_{d \,|\, |G|} n_d(G) \cdot \varphi(d),$$

where φ is the Euler φ-function.

Proof:

This is clear since every element of G is a generator of a cyclic subgroup and there are exactly $\varphi(d)$ different generators for a cyclic group of order d.

Q.E.D.

Remark 15. *When G is cyclic of cardinality n, we get back the well-known formula*

$$n = \sum_{d \,|\, n} \varphi(d).$$

Theorem 4.25. *Let G be a finite abelian group, then the number of minimal cocyclic subgroups is the same as the number of maximal cyclic subgroups.*

Proof:

This follows again by duality theory for finite abelian groups.

Q.E.D.

Theorem 4.26. *Let G_1 and G_2 be two finite abelian groups and suppose that their cardinalities are coprime. For any subgroup H of $G_1 \times G_2$, there exist subgroups H_i of G_i such that*

$$H = H_1 \times H_2.$$

Proof:

This follows from the following fact. Suppose G is a finite abelian group, $H \leq G$ and let $g_1, g_2 \in G$ be such that their orders s_1 and s_2 are relatively prime. If $g_1 \cdot g_2 \in H$, then $g_1, g_2 \in H$ necessarily. Indeed

$$(g_1 \cdot g_2)^{s_2} = g_1^{s_2} \in H.$$

Since $(s_1, s_2) = 1$, there exist $m, n \in \mathbb{Z}$ such that $ms_1 + ns_2 = 1$, in which case

$$g_1 = g_1^{ms_1+ns_2} = (g_1^{s_2})^n \in H,$$

and by a similar argument, one concludes that $g_2 \in H$ as well.

Thus, if $h = (h_1, h_2)$ for some $h_i \in G_i$, then since $h = (h_1, 0) + (0, h_2)$ and the order of $(h_1, 0)$ and of $(0, h_2)$ are relatively prime, we conclude that $(h_1, 0), (0, h_2) \in H$. This shows that $H = H_1 \times H_2$, where $H_i = \pi_i(H)$ and the π_i $(i = 1, 2)$ are the natural projection maps.

Q.E.D.

Corollary 4.27. *Let G_1 and G_2 be two finite abelian groups. If $G = G_1 \times G_2$, where $(|G_1|, |G_2|) = 1$, every cyclic subgroup of G is of the form $H_1 \times H_2$, where H_i is a cyclic subgroup of G_i. Thus*

$$n_G = n_{G_1} \cdot n_{G_2}.$$

Also, we have that every minimal cocyclic subgroup of G is of the form $H_1 \times H_2$, where H_i is a minimal cocyclic subgroup of G_i; therefore,

$$n'_G = n'_{G_1} \cdot n'_{G_2}.$$

Proof:

The first part of the corollary about the cyclic subgroups is clear. Let H be a minimal cocyclic subgroup. By Theorem 4.26, we know that $H = H_1 \times H_2$ for some subgroups $H_i \subseteq G_i$. Since

$$G/H \simeq G_1/H_1 \times G_2/H_2,$$

and G/H is cyclic, we see that H_i is a cocyclic subgroup of G_i for $i = 1, 2$. Suppose that H_1 is not minimal cocyclic, then there exists $K_1 \subsetneq H_1$ such that G_1/K_1 is cyclic.

But then $K_1 \times H_2$ would be cocyclic as well which would be a contradiction with the minimality of $H_1 \times H_2$. A similar argument shows that H_2 is also a minimal cocyclic subgroup of G_2. As for the formula

$$n'_G = n'_{G_1} \cdot n'_{G_2},$$

we just have to show that given minimal cocyclic subgroups $H_i \subseteq G_i$, the product $H_1 \times H_2$ is a minimal cocyclic subgroup of $G_1 \times G_2$. But this follows again from Theorem 4.26.

Q.E.D.

Remark 16. *These two last results clearly extend by induction to a finite product of finite abelian groups whose cardinalities are coprime to each other.*

Theorem 4.28. *Suppose that $G \simeq (\mathbb{Z}/p\mathbb{Z})^n$, then we have*

$$n'_G = n_p(G) = 1 + p + \ldots + p^{n-1} = \frac{p^n - 1}{p - 1}.$$

Proof:

The group G is a p-elementary abelian group, thus we can view G as a $\mathbb{F}_p$-vector space of dimension n. It is clear that the minimal cocyclic subgroups corresponds to the subgroups of index exactly p in G. Hence we want to count the number of hyperplanes in G when viewed as a $\mathbb{F}_p$-vector space. The number of families of $n-1$ linearly independent vectors in G is equal to

$$(p^n - 1)(p^n - p) \cdot \ldots \cdot (p^n - p^{n-2}),$$

whereas given a $\mathbb{F}_p$-vector space of dimension $n-1$ there are

$$(p^{n-1} - 1)(p^{n-1} - p) \cdot \ldots \cdot (p^{n-1} - p^{n-2})$$

different choices of basis. Therefore, there are

$$\frac{(p^n - 1)(p^n - p) \cdot \ldots \cdot (p^n - p^{n-2})}{(p^{n-1} - 1)(p^{n-1} - p) \cdot \ldots \cdot (p^{n-1} - p^{n-2})} = \frac{p^n - 1}{p - 1}$$

hyperplanes in G.

Q.E.D.

4.5 An explicit example

In this section, we look at a particular example which is given in [17]. Let p, q be two odd prime numbers satisfying

$$p \equiv 1 \pmod 4, \quad q \equiv 3 \pmod 4, \quad \text{and} \quad \left(\frac{p}{q}\right) = 1. \tag{4.4}$$

Here is a table containing the pairs (p, q) for p and q below 100.

Table 4.1: Pairs of prime numbers (p, q) satisfying properties (4.4).

q \ p	5	13	17	29	37	41	53	61	73	89	97
3		X			X			X	X		X
7				X	X		X				
11	X				X		X			X	X
19	X		X					X	X		
23		X		X		X			X		
31	X					X					X
43		X	X			X	X				X
47			X		X		X	X		X	X
59	X		X	X		X	X				
67			X	X	X				X	X	
71	X			X	X				X	X	
79	X	X							X	X	X
83			X	X	X	X		X			

Let $K' = \mathbb{Q}(\zeta_q)^{D_p}$, where D_p is the decomposition group associated to p in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. We remark that $\mathbb{Q}(\sqrt{-q}) \subseteq K'$. Let K'^{+} be the maximal real subfield of K'. Let l be an odd prime number, different from p and q and satisfying

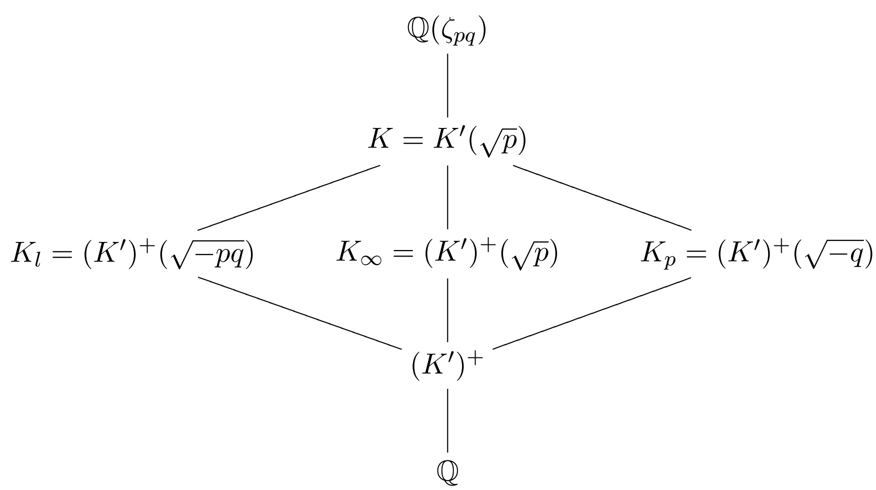
$$\left(\frac{K'/\mathbb{Q}}{l}\right) \neq 1, \quad \left(\frac{K'^{+}/\mathbb{Q}}{l}\right) = 1, \text{ and } \left(\frac{p}{l}\right) = -1,$$

(i.e. l splits completely in $K'^{+}/\mathbb{Q}$, but does not split completely in $K'/\mathbb{Q}$ nor in $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$). Let $K = K'(\sqrt{p})$. Then $S = \{\infty, p, q, l\}$ is a 1-cover for $\widehat{G}$ and $S_{min} =$

$\{\infty, p, l\}$. Indeed, we have

$$\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \times \mathrm{Gal}(K'/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z},$$

where $m = (q-1)/2f_p$, and f_p is the inertia index of p is $\mathbb{Q}(\zeta_q)$. Since $q \equiv 3 \pmod 4$ we have $(m, 2) = 1$. Therefore, we have by Theorem 4.28 and Corollary 4.27 that the minimal cocyclic subgroups are precisely the 3 subgroups of cardinality 2 of G. We then use Theorem 4.19 and Theorem 4.20 in order to conclude that $S = \{\infty, p, q, l\}$ is a 1-cover for $\widehat{G}$ and $S_{min} = \{\infty, p, l\}$. The following diagram of fields might be useful in order to see what is going on.



In the diagram, we use the following notation. If v is a place of $\mathbb{Q}$, then K_v is the fixed field of K fixed by the decomposition group G_v (not to be confused with a completion of K at some place).

The extended abelian rank one Stark conjecture is still unknown for this example. Among all the different possibilities, luckily there are some which are biquadratic and this gives a way to see what is happening. In fact, almost all of the pairs in the table above correspond to biquadratic examples. There are only 12 non-biquadratic examples in the table and they are the following ones:

$$(5, 31), (5, 71), (29, 7), (29, 67), (37, 67), (37, 71), (41, 43),$$

and

$$(89, 67), (89, 79), (97, 31), (97, 43), (97, 79).$$

By the way, the extended abelian rank one Stark conjecture is known to be true for all biquadratic extensions, see Theorem 6.9 in [19]. In fact, a stronger statement is true as explained in Chapter 5, Theorem 5.25.

Nevertheless, it seems appropriate to work out a biquadratic example explicitly. Let us take $p = 37$, $q = 7$ and $l = 5$. In this case, we have $K = \mathbb{Q}(\sqrt{-7}, \sqrt{37})$, $K_5 = \mathbb{Q}(\sqrt{-7 \cdot 37})$, $K_\infty = \mathbb{Q}(\sqrt{37})$, and $K_{37} = \mathbb{Q}(\sqrt{-7})$. We already computed the Stark units for the abelian rank one Stark conjecture for the two fields K_5 and K_{37} at the end of Section 3.5.2. We found that a Stark unit for $St(K_5/\mathbb{Q}, \{\infty, 5, 7, 37\}, 5)$ is given by

$$\varepsilon_5 = \frac{13 + 3\sqrt{-7 \cdot 37}}{13 - 3\sqrt{-7 \cdot 37}} = \left(\frac{13 + 3\sqrt{-7 \cdot 37}}{50}\right)^2.$$

Similarly, a Stark unit for $St(K_{37}/\mathbb{Q}, \{\infty, 7, 37\}, 37)$ is given by

$$\frac{3 + 2\sqrt{-7}}{3 - 2\sqrt{-7}}.$$

Thus, using Theorem 3.10, we find that a Stark unit for $St(K_{37}/\mathbb{Q}, \{\infty, 5, 7, 37\}, 37)$ is

$$\varepsilon_{37} = \left(\frac{3 + 2\sqrt{-7}}{\sqrt{37}}\right)^4,$$

since the Frobenius automorphism associated to the prime 5 in K_{37} is the unique non-trivial automorphism of this quadratic extension. Since both ∞ and 7 split completely in K_∞ we have that a Stark unit for this extension is simply

$$\varepsilon_\infty = 1,$$

by the last remark following Conjecture 3.7. We let

$$\eta_5 = \frac{13 + 3\sqrt{-7 \cdot 37}}{50}, \ \eta_{37} = \left(\frac{3 + 2\sqrt{-7}}{\sqrt{37}}\right)^2, \text{ and } \eta_\infty = 1.$$

We have further $w_5 = w_{37} = w_\infty = 2$ and it is clear that both η_{37} and η_∞ are 2-abelian over $\mathbb{Q}$. We are just left to show that η_5 is 2-abelian as well and we use Theorem A.5 for this purpose. We let

$$n_1 = n_\sigma = 1, \quad \alpha_1 = 1, \text{ and } \alpha_\sigma = \frac{50}{13 + 3\sqrt{-7 \cdot 37}},$$

where σ is the unique non-trivial automorphism of $\mathbb{Q}(\sqrt{-7\cdot 37})/\mathbb{Q}$. One has trivially

$$\alpha_1^{\sigma-1} = \alpha_\sigma^{1-1} = 1,$$

and a simple computation shows that

$$\eta_5^{\sigma-1} = \alpha_\sigma^2.$$

Thus, we conclude that $ESt(\mathbb{Q}(\sqrt{-7},\sqrt{37})/\mathbb{Q},\{\infty,5,7,37\})$ is true using the idea exposed after Proposition 4.13.

So far, we have not been able to prove that the extended abelian rank one Stark conjecture is always verified for this example, but in the case where $p\equiv 1 \pmod q$ we can prove the existence of a Stark unit (but not the abelian condition). If $p\equiv 1 \pmod q$, then $D_p = 1$, $K' = \mathbb{Q}(\zeta_q)$, and $K_p = \mathbb{Q}(\zeta_q)$. The following pairs (p,q) in the table above satisfy this extra property:

$$(13,3),(29,7),(37,3),(61,3),(73,3),(89,11), \text{ and } (97,3).$$

We try to use the idea exposed after Proposition 4.13 again. Therefore, we have to compute the usual Stark units for each of the three fields K_l, K_∞, and K_p.

Let us start at K_p which is the full cyclotomic field $\mathbb{Q}(\zeta_q)$ under our assumptions. Formula (3.7) tells us that a Stark unit for $St(\mathbb{Q}(\zeta_q)/\mathbb{Q},\{\infty,q,p\},p)$ is given by

$$\varepsilon = \left(\frac{g_q(\mathfrak{p})}{\sqrt{p}}\right)^{2q},$$

where $\mathfrak{p}$ is a prime of $\mathbb{Q}(\zeta_q)$ lying above p. Using Theorem 3.10, we see that a Stark unit for $St(\mathbb{Q}(\zeta_q)/\mathbb{Q},\{\infty,q,p,l\},p)$ is given by

$$\varepsilon_p = \varepsilon^{1-\sigma_l^{-1}},$$

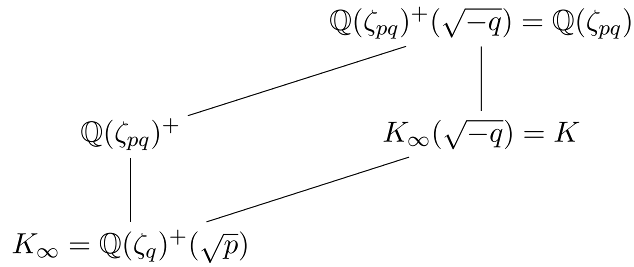
where σ_l is the Frobenius automorphism associated to l in the extension $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. In fact, since everything lives in $\mathbb{Q}(\zeta_{pq})$ we can view σ_l as being the Frobenius automorphism in this last extension $\mathbb{Q}(\zeta_{pq})/\mathbb{Q}$, and this is what we shall do from now on. Thus, we let

$$\eta_p = \left(\left(\frac{g_q(\mathfrak{p})}{\sqrt{p}}\right)^q\right)^{1-\sigma_l^{-1}} \in K = \mathbb{Q}(\zeta_q)(\sqrt{p}).$$

Remark for future use that in fact

$$\eta_p \in K_p = \mathbb{Q}(\zeta_q).$$

As for K_∞, we look at the following diagram of fields.



We know, see Section 3.4.1, that a Stark unit for $\mathbb{Q}(\zeta_{pq})^+/\mathbb{Q}$ and $S = \{\infty, p, q\}$ is given by

$$\varepsilon = (1 - \zeta_{pq})(1 - \zeta_{pq}^{-1}) = -\zeta_{pq}^{-1}(1 - \zeta_{pq})^2,$$

and because of Theorem 3.25, a Stark unit for K_∞ is given by

$$\varepsilon' = N_{\mathbb{Q}(\zeta_{pq})^+/K_\infty}(\varepsilon).$$

We want to show that this last algebraic number is a square in K. We remark that we have an isomorphism

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{pq})/K) \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}(\zeta_{pq})^+/K_\infty),$$

given by the usual restriction map. For each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{pq})^+/K_\infty)$, we choose a lift $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}(\zeta_{pq})/K)$ and we denote the elements of the first Galois group by $\sigma_1, \ldots, \sigma_s$, where $s = (p-1)/2$. Thus, we have

$$\begin{aligned}
\varepsilon' &= \varepsilon^{\sigma_1 + \ldots + \sigma_s} \\
&= \varepsilon^{\tilde{\sigma}_1 + \ldots + \tilde{\sigma}_s} \\
&= (-1)^s (\alpha^{\tilde{\sigma}_1 + \ldots + \tilde{\sigma}_s})^2 \\
&= (\alpha^{\tilde{\sigma}_1 + \ldots + \tilde{\sigma}_s})^2,
\end{aligned}$$

since s is even and where

$$\alpha = \zeta_{pq}^{\frac{pq-1}{2}}(1-\zeta_{pq}).$$

Again by Theorem 3.10, a Stark unit we are looking for is given by

$$\varepsilon_\infty = (\varepsilon')^{1-\sigma_l^{-1}},$$

and we let

$$\eta_\infty = \left(\alpha^{\tilde{\sigma}_1+\ldots+\tilde{\sigma}_s}\right)^{1-\sigma_l^{-1}}.$$

We now have to write down a Stark unit for $St(K_l/\mathbb{Q}, \{\infty, p, q, l\}, l)$.

Lemma 4.29. *We have* $w_{K_l} = 2$.

Proof:

The only possibilities are 2, $2p$, $2q$ or $2pq$. We proceed by elimination. If $w_{K_l} = 2pq$ then $K_l \subseteq \mathbb{Q}(\zeta_{pq}) \subseteq K_l$ and comparing degrees, we would have

$$q-1 = (p-1)(q-1),$$

which is impossible. If $w_{K_l} = 2q$, then $\mathbb{Q}(\zeta_q) \subseteq K_l$ and comparing degrees, we would have an equality $K_l = \mathbb{Q}(\zeta_q)$. But l does not split completely in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$, hence this case is impossible as well. At last, if $w_{K_l} = 2p$, then we would have $\mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(\zeta_p) \subseteq K_l$, but again l does not split completely in $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$. We conclude that $w_{K_l} = 2$.

Q.E.D.

We shall denote the usual inclusion of ideals $I_{K_l} \longrightarrow I_{\mathbb{Q}(\zeta_{pq})}$ by the letter ι. Let $\mathfrak{l}$ be a prime ideal of K_l lying above l. Proposition 3.27 implies that a Stark unit for $St(K_l/\mathbb{Q}, \{\infty, p, q, l\}, l)$ is given by $\varepsilon_l = \tau(\iota(\mathfrak{l}))^2$. Thus, we let

$$\eta_l = \tau(\iota(\mathfrak{l})).$$

So far, we only know that $\eta_l \in K_l(\zeta_l)$, we will show in the next series of lemmata that in fact $\eta_l \in K_l$.

Lemma 4.30. *The inertia index of* l *in* $\mathbb{Q}(\zeta_{pq})$ *is even.*

Proof:

Indeed, l splits completely in $(K')^+$, but not in $K' = \mathbb{Q}(\zeta_q)$ and is unramified there as well. Hence, the inertia index of l in this extension is precisely 2. Using the usual formulae relating the inertia index for the top field to the ones of intermediate fields, we conclude that the inertia index of l in $\mathbb{Q}(\zeta_{pq})$ is even.

Q.E.D.

Lemma 4.31. *Given any ideal $\mathfrak{L}$ of $\mathbb{Q}(\zeta_{pq})$ lying above l, we have*

$$\tau(\mathfrak{L}) \in \mathbb{Q}(\zeta_{pq}).$$

Proof:

Let t be an integer such that $(t, pql) = 1$ and $t \equiv 1 \pmod{pq}$. We then have

$$G(\chi_{\mathfrak{L}})^{\sigma_t} = \overline{\chi_{\mathfrak{L}}(t)} G(\chi_{\mathfrak{L}}),$$

because of Theorem 2.4. We view t as being in the prime field $\mathbb{F}_l$ and we let γ be a generator (primitive element) of $(\mathbb{Z}[\zeta_{pq}]/\mathfrak{L})^\times$. Write $t = \gamma^a$ for some integer a. If we show that $pq \,|\, a$, then we would be done, because in that case

$$\chi_{\mathfrak{L}}(t) = \chi_{\mathfrak{L}}(\gamma^a) = \chi_{\mathfrak{L}}(\gamma)^a = 1,$$

since $\chi_{\mathfrak{L}}$ has order pq. This is indeed the case, since

$$\operatorname{ord}(\gamma^a) = \frac{l^f - 1}{(l^f - 1, a)} \,\Big|\, (l-1),$$

and both p and q divide $l^f - 1$ but not $l - 1$. This last statement is true because otherwise, it would contradict the splitting condition imposed on l in either $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ or in $K = \mathbb{Q}(\zeta_q)/\mathbb{Q}$. This shows that $G(\chi_{\mathfrak{L}}) \in \mathbb{Q}(\zeta_{pq})$ and we can conclude that the same is true for $\tau(\mathfrak{L})$ because of Lemma 4.30.

Q.E.D.

Lemma 4.32. *Given a prime ideal $\mathfrak{l}$ of K_l lying above l, we have $\tau(\iota(\mathfrak{l})) \in K_l$.*

Proof:

This follows immediately from Lemma 4.31 and Theorem 2.17.

Q.E.D.

We are thus left to show that η_p is $2q$-abelian, and that both η_∞ and η_l are 2-abelian over $\mathbb{Q}$. Let us start with η_p. We have that $l \equiv -1 \pmod q$ and therefore σ_l acts as complex conjugation on $\mathbb{Q}(\zeta_q)$. Moreover, σ_l acts on $\mathbb{Q}(\sqrt{p})$ as $\sqrt{p} \mapsto -\sqrt{p}$ since $\left(\frac{p}{l}\right) = -1$. Therefore,

$$\eta_p = \frac{\left(\frac{G(\chi_\mathfrak{p})}{\sqrt{p}}\right)^q}{\left(\frac{\overline{G(\chi_\mathfrak{p})}}{-\sqrt{p}}\right)^q} = \left(-\frac{G(\chi_\mathfrak{p})}{\overline{G(\chi_\mathfrak{p})}}\right)^q = \left(-\frac{G(\chi_\mathfrak{p})^2}{|G(\chi_\mathfrak{p})|^2}\right)^q = \left(i\frac{G(\chi_\mathfrak{p})}{\sqrt{p}}\right)^{2q},$$

the last equality being true because of Theorem 2.6. We can then conclude that η_p is $2q$-abelian, since $i\frac{G(\chi_\mathfrak{p})}{\sqrt{p}}$ is clearly abelian over $\mathbb{Q}$.

Unfortunately, we were not able to prove that η_∞ and η_l are 2-abelian over $\mathbb{Q}$ using the following lemma.

Lemma 4.33. *Let K/k be a finite cyclic extension with Galois group $G =< \sigma >$ and let $\alpha \in K^\times$. Suppose also that we are given $n_\sigma \in \mathbb{Z}$ such that $\zeta^\sigma = \zeta^{n_\sigma}$. Then the extension $K(\alpha^{1/w_K})/k$ is abelian if and only if*

$$\alpha^{\sigma - n_\sigma} \in (K^\times)^{w_K}.$$

Proof:

The necessary part is clear by Theorem A.5. Conversely, suppose that

$$\alpha^{\sigma - n_\sigma} \in \alpha_\sigma^{w_K},$$

for some $\alpha_\sigma \in K^\times$. Given $n \in \mathbb{Z}$, note that in $\mathbb{Z}[G]$ we have the following identity

$$(\sigma^i - n^i) = (\sigma - n)(n^{i-1} + n^{i-2}\sigma + \ldots + \sigma^{i-1}), \tag{4.5}$$

for all $i = 1, \ldots, g$, where $g = |G|$. For $i = 1, \ldots, g-1$, set

$$\alpha_{\sigma^i} = \alpha_\sigma^{n_\sigma^{i-1} + n_\sigma^{i-2}\sigma + \ldots + \sigma^{i-1}},$$

and

$$n_{\sigma^i} = n_\sigma^i.$$

Using identity (4.5), it is simple to check that the conditions of Theorem A.5 are satisfied.

Q.E.D.

Chapter 5

Investigation of the extended abelian rank one Stark conjecture

As we explained previously, the formulae of Propositions 4.13 and 4.16 are sometimes useful in order to show that Conjecture 4.11 and 4.15 hold true. The best thing one could hope for is that either one of the conjectures would be "provable one place $v \in S_{min}$ at the time". This will lead us to formulate a stronger question than the original one. Before we carry on with this task, here are some observations.

In order to detect where exactly the various Stark units are powers, it might be useful to consider the following map. If K/L is a finite extension of number fields and n any positive integer, we shall study a little

$$\varphi : L^{\times}/(L^{\times})^n \longrightarrow K^{\times}/(K^{\times})^n,$$

induced by the natural inclusion $L^{\times} \to K^{\times}$.

Proposition 5.1. *Let K/L be a Galois extension of number fields and let n be a positive integer satisfying $(n, w_K) = 1$. Then the group homomorphism*

$$\varphi : L^{\times}/(L^{\times})^n \longrightarrow K^{\times}/(K^{\times})^n,$$

is injective.

Proof:

Let $x \in L^{\times}$ and suppose that $x = y^n$ for some $y \in K^{\times}$. Let $\sigma \in \mathrm{Gal}(K/L)$ and consider

$y^{\sigma-1}$. If we raise it to the power n, we get

$$(y^{\sigma-1})^n = x^{\sigma-1} = 1.$$

In other words, $y^{\sigma-1} \in \mu_n \cap K^\times$. Since $(n, w_K) = 1$, we have $\mu_n \cap K^\times = \{1\}$. Therefore, we get $y^{\sigma-1} = 1$ for all $\sigma \in \mathrm{Gal}(K/L)$ and we conclude that $y \in L^\times$.

Q.E.D.

Suppose that we are in the situation of Conjecture 4.4 for an abelian extension K/k and a 1-cover S. Let $\mathfrak{p} \in S_{min}$ be a finite prime, $n = |G_\mathfrak{p}|$, and $L = K^{G_\mathfrak{p}}$. Let ε be a Stark unit for $St(L/k, S, \mathfrak{p})$ and suppose also that it is a n-th power in K, that is there exists $\eta \in K^\times$ such that $\eta^n = \varepsilon$. Let us make the further hypothesis that $(n, w_K) = 1$ (such examples exist, see B.2.2). In this case, the last proposition shows that $\eta \in L^\times$. Moreover it is also an S-unit. This simply means that at times, if a Stark unit for L/k is a certain power in K then it has no choice but being a n-th power in L to begin with.

Here is a remark concerning the abelian condition. Suppose that we have a tower of fields $k \subseteq L \subseteq K$ such that K/k is abelian and let $x \in L^\times$. Then the extension $K(x^{\frac{1}{w_L}})/k$ is abelian if and only if $L(x^{\frac{1}{w_L}})/k$ is abelian. This is clear since $K(x^{1/w_L})$ is the compositum $K \cdot L(x^{1/w_L})$.

Remark also that if we introduce a finite set of finite primes T satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$, then the map

$$E_{L,S,T}/E_{L,S,T}^n \longrightarrow E_{K,S,T}/E_{K,S,T}^n$$

is always injective for any choice of n whenever K/L is Galois by an argument similar to the one given in the proof of Proposition 5.1.

5.1 A stronger question (the S-version)

Notation 1. Throughout this section, we shall be dealing with an abelian extension K/k of number fields and a single prime $v \in S_{min}$, where S is a 1-cover for K/k containing $S(K/k)$. We shall denote $|G_v|$ by n and K^{G_v} by L. Moreover the quotient group $\Gamma_v = G/G_v$ will be referred to as Γ. This will simplify the writing quite a bit and should not confuse the reader.

We will also have to deal with sets S of places of k satisfying the following properties:

Hypothesis 5.1.

(1) *$S \supseteq S(K/k)$,*

(2) *S is a 1-cover for K/k,*

(3) *$S \neq S_{min}$.*

Question 5.2. *Let K/k be an abelian extension of number fields and let S be a finite set of places satisfying Hypothesis 5.1. Fix $v \in S_{min}$ and $w \in S_L$ lying above v. Does there exist $\eta \in (\tilde{E}_{L,S})_{1,S}$ such that*

$$\theta'_{L/k,S}(0) = \frac{n}{w_L} \cdot R_{L/k,w}(\eta)?$$

Moreover, if $u \in E_{L,S}$ satisfies $\tilde{u} = \eta$, is the extension $L(u^{\frac{1}{w_L}})/k$ abelian?

We make some remarks:

(1) As in Conjecture 3.39, the uniqueness of η follows because we require $\eta \in (\tilde{E}_{L,S})_{1,S}$. If $|S| \geq 3$, we could replace this condition and require instead that $|\eta|_{w'} = 1$ for all $w' \in S_L$ not lying above v.

(2) We shall denote this question by $St(K/k, S, v)$, where $v \in S_{min}$, or $St(K/k, S, v, w)$ if we want to specify the choice of $w \in S_L$. The truth of the question is independent of the choice of w (by an argument similar to the one given for the usual abelian rank one Stark conjecture, see the remarks following Conjecture 3.7), but the unit η does depend on the choice of w in a simple way.

(3) Moreover, in the case of an affirmative answer, the $\eta \in (\tilde{E}_{L,S})_{1,S}$ satisfying

$$\theta'_{L/k,S}(0) = \frac{n}{w_L} \cdot R_{L/k,w}(\eta),$$

will be called a Stark n-unit.

(4) If S contains a split prime v then either $S_{min} = \varnothing$ or $S_{min} = \{v\}$. In the latter case, $St(K/k, S, v)$ is just the usual abelian rank one Stark conjecture, since $K = L$ and $n = 1$ in that case.

(5) Let $u_1, u_2 \in E_{L,S}$ be such that $\tilde{u}_1 = \eta$ and $\tilde{u}_2 = \varepsilon$, where ε is the usual Stark unit for $St(L/k, S, v, w)$ that is

$$\theta'_{L/k,S}(0) = \frac{1}{w_L} \cdot R_{L/k,w}(\varepsilon).$$

We then have $u_2 = \zeta u_1^n$ for some $\zeta \in \mu_L$, since $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-module when restricted to the $()_{1,S}$ subspace. In other words, one of the Stark units, namely $\zeta^{-1}u_2$, is an n-th power of u_1. If one intends to prove that Question 5.2 has an affirmative answer by showing that the usual Stark unit in L is a power of some other unit, than the determination of the argument of a Stark unit becomes important.

Proposition 5.3. *Let K/k be an abelian extension of number fields and let S be a finite set of places satisfying Hypothesis 5.1. If $St(K/k, S, v)$ is true for all $v \in S_{min}$, then the extended abelian rank one Stark conjecture (Conjecture 4.11) is true.*

Proof:

This is a direct consequence of Proposition 4.13 and the remark following it.

Q.E.D.

5.1.1 The case where v is a finite prime

Suppose we are in the setting of the question $St(K/k, S, v)$ where S is a 1-cover for K/k and $v \in S_{min}$. In this section, we derive some consequences in the case where v is a finite prime. Because of that, we switch of notation, and we denote v by $\mathfrak{p}$. Fix a prime $\mathfrak{P}$ of $L = K^{G_\mathfrak{p}}$ lying above $\mathfrak{p}$ and denote $S \smallsetminus \{\mathfrak{p}\}$ by $R_\mathfrak{p}$ or R when $\mathfrak{p}$ is fixed. Moreover, as before $n = n_\mathfrak{p} = |G_\mathfrak{p}|$, and $\Gamma = \Gamma_\mathfrak{p} = G/G_\mathfrak{p}$. Since

$$\theta'_{L/k,S}(s) = \frac{\sigma_\mathfrak{p}^{-1}}{\mathbb{N}(\mathfrak{p})^s} \cdot \log \mathbb{N}(\mathfrak{p}) \cdot \theta_{L/k,R}(s) + \left(1 - \frac{\sigma_\mathfrak{p}^{-1}}{\mathbb{N}(\mathfrak{p})^s}\right) \cdot \theta'_{L/k,R}(s),$$

by Lemma 3.17 and $\mathfrak{p}$ splits in L/k (i.e. $\sigma_\mathfrak{p} = 1$), one gets

$$\theta'_{L/k,S}(0) = \log \mathbb{N}(\mathfrak{p}) \cdot \theta_{L/k,R}(0). \tag{5.1}$$

Now, if $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ is true, there exists $\eta \in (\tilde{E}_{L,S})_{1,S}$ satisfying

$$\begin{aligned}\theta'_{L/k,S}(0) &= \frac{n}{w_L} \cdot R_{L/k,\mathfrak{P}}(\eta) \\ &= -\frac{n}{w_L} \cdot \sum_{\gamma \in \Gamma} \log |\eta|_{\mathfrak{P}^\gamma} \cdot \gamma \\ &= \frac{n}{w_L} \cdot \sum_{\gamma \in \Gamma} \mathrm{ord}_{\mathfrak{P}^\gamma}(\eta) \cdot \log \mathbb{N}(\mathfrak{p}) \cdot \gamma.\end{aligned}$$

Hence, combining this last line with equation (5.1), one gets

$$\theta_{L/k,R}(0) = \frac{n}{w_L} \cdot \sum_{\gamma \in \Gamma} \mathrm{ord}_{\mathfrak{P}^\gamma}(\eta) \cdot \gamma.$$

Therefore, one should have

$$\frac{w_L \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]. \tag{5.2}$$

Moreover, we get the following equality of ideals in L:

$$\mathfrak{P}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (\eta). \tag{5.3}$$

Combining (5.2) and (5.3), we arrive at the following conclusion.

Proposition 5.4. *Let K/k be an abelian extension of number fields, S be a finite set of places satisfying Hypothesis 5.1 and suppose that $|S| \geq 3$. Let $\mathfrak{p}$ be a finite prime in S_{min}. With the same notation as above, if $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ is true with Stark n-unit $\eta = \tilde{u}$ then the following statements hold:*

(1) $\frac{w_L \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) $\mathfrak{P}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (u)$ *as ideals in L,*

(3) *The extension $L(u^{\frac{1}{w_L}})/k$ is abelian,*

(4) $u \in L^0$, *i.e. it is an anti-unit.*

Conversely, if (1) holds and if there exists $u \in L^\times$ for which (2),(3), and (4) hold then $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ holds true with Stark n-unit $\eta = \tilde{u}$.

Proof:

Just reverse the steps before the proposition.

Q.E.D.

5.1.2 An extension of the Brumer-Stark conjecture

In this section, not only do we assume that $\mathfrak{p} \in S_{min}$ is a finite prime, but we suppose further that $\mathfrak{p}$ is unramified.

Definition 5.5. Let K/k be an abelian extension and let R be a finite set of primes containing $S(K/k)$. Let $\mathfrak{p} \notin R$ be any finite prime, necessarily unramified. The set R is called a $\mathfrak{p}$-1-cover if $S = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k, $\mathfrak{p} \in S_{min}$, and $S \neq S_{min}$.

Remark 17. *A* $\mathfrak{p}$-1-*cover is not necessarily a* 1-*cover.*

Definition 5.6. Let K/k be an abelian extension of number fields and R a finite set of places containing $S(K/k)$ which is a $\mathfrak{p}$-1-cover for K/k. We define $A = A_{R,L}$ to be the subgroup of Cl_L generated by the $[\mathfrak{Q}]$ where $\mathfrak{Q}$ is a prime ideal of L lying above a prime $\mathfrak{q} \notin R$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Moreover, define $I_L^* = I_{L,R}^*$ to be the subgroup of I_L satisfying $I_L^*/P_L = A$.

Question 5.7. *With the notation as above, are the following statements true if* $|R| \geq 2$*?*

(1) $\frac{w_L \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) *For any* $\mathfrak{a} \in I_L^*$, *there exists* $\eta_R(\mathfrak{a}) \in L^\times$ *satisfying*

$$\mathfrak{a}^{\frac{w_L \cdot \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{a})),$$

as ideals in L,

(3) $\eta_R(\mathfrak{a}) \in L^0$,

(4) *The extension* $L(\eta_R(\mathfrak{a})^{\frac{1}{w_L}})/k$ *is abelian.*

We make some remarks.

(1) We shall denote this question by $BrSt(K/k, R)$, where R is a given $\mathfrak{p}$-1-cover.

(2) Suppose that R is any finite set of places containing $S(K/k)$ and $\mathfrak{p} \notin R$ is a place which splits completely in K/k. The set $S = R \cup \{\mathfrak{p}\}$ is automatically a 1-cover. If $S_{min} \neq \varnothing$, then $S_{min} = \{\mathfrak{p}\}$ necessarily and we get back the original Brumer-Stark conjecture.

Lemma 5.8. *Let R be a $\mathfrak{p}$-1-cover for K/k. Let $\mathfrak{q}$ be any finite unramified prime such that*

(1) $\mathfrak{q} \notin R$;

(2) $\left(\frac{K/k}{\mathfrak{q}}\right) \in G_{\mathfrak{p}}$.

Then $S = R \cup \{\mathfrak{q}\}$ is a 1-cover for K/k. Moreover, $\mathfrak{q} \in S_{min}$.

Proof:

Since R is a $\mathfrak{p}$-1-cover, we know that $S' = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k and also $\mathfrak{p} \in S'_{min}$. The fact that S' is a 1-cover implies that every non-trivial character has at least one decomposition group in its kernel, and the fact that $\mathfrak{p} \in S'_{min}$ shows that there exists a non-trivial character $\chi_0 \in \widehat{G}_{1,S}$ such that $G_{\mathfrak{p}}$ is the unique decomposition group contained in $\mathrm{Ker}(\chi_0)$. If we replace S' by $S = R \cup \{\mathfrak{q}\}$, we see that it is still a 1-cover and also $G_{\mathfrak{q}}$ is the unique decomposition group contained in $\mathrm{Ker}(\chi_0)$, that is $\mathfrak{q} \in S_{min}$.

Q.E.D.

Proposition 5.9. *Let R be a $\mathfrak{p}$-1-cover, $L = K^{G_{\mathfrak{p}}}$ and suppose that Gross's conjecture is true for $K/L/k$. Then $BrSt(K/k, R)$ is equivalent to $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$ for all finite primes $\mathfrak{q} \notin R$ satisfying*

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Proof:

Remark first that by Lemma 5.8, it makes sense to talk about $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$. Suppose first that $BrSt(K/k, R)$ holds and let $\mathfrak{q} \notin R$ be a prime ideal of k satisfying the condition of the proposition. Let $\mathfrak{Q}$ be an ideal of $L = K^{G_{\mathfrak{p}}}$ lying above $\mathfrak{q}$. Note that $\mathfrak{Q} \in I_L^*$ and thus $BrSt(K/k, R)$ implies that there exists $\eta_R(\mathfrak{Q}) \in L^0$ satisfying

$$\mathfrak{Q}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{Q})),$$

and such that $L(\eta_R(\mathfrak{Q})^{\frac{1}{w_L}})/k$ is abelian. We can now conclude by Proposition 5.4.

Conversely, suppose that $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$ is true for all $\mathfrak{q} \notin R$ satisfying the condition above. Let $\mathfrak{a} \in I_L^*$, then $[\mathfrak{a}] \in A$. Now, by the definition of A we have

$$[\mathfrak{a}] = C_1^{r_1} \cdot \ldots \cdot C_s^{r_s},$$

where, for all $i = 1, \dots, s$, there exists a prime $\mathfrak{Q}_i$ of L such that $[\mathfrak{Q}_i] = C_i$. Moreover, $\mathfrak{Q}_i$ lies above another prime $\mathfrak{q}_i$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}_i}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

There exists $\alpha \in L^\times$ such that $\mathfrak{a} = \alpha \cdot \mathfrak{Q}_1^{r_1} \cdot \ldots \cdot \mathfrak{Q}_s^{r_s}$. Now, $St(K/k, R \cup \{\mathfrak{q}_i\}, \mathfrak{q}_i, \mathfrak{Q}_i)$ implies that there exists $\eta_R(\mathfrak{Q}_i) \in L^0$ which satisfies

$$\mathfrak{Q}_i^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{Q}_i)),$$

and such that $L(\eta_R(\mathfrak{Q}_i)^{\frac{1}{w_L}})/k$ is abelian. Therefore, we have

$$\mathfrak{a}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = \alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}} \cdot \prod_{i=1}^{s} \eta_R(\mathfrak{Q}_i)^{r_i},$$

and we just have to set

$$\eta_R(\mathfrak{a}) = \alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}} \cdot \prod_{i=1}^{s} \eta_R(\mathfrak{Q}_i)^{r_i},$$

in order to conclude. Indeed,

$$\alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}} \in L^0,$$

by Lemma 3.21 below and is w_L-abelian over k by Lemma 3.22 below combined with Theorem 5.29 (Theorem 5.29 implies that the second condition of Lemma 3.22 is satisfied. This is where we use Gross's conjecture).

Q.E.D.

5.2 The (S, T)-version of the stronger question

The purpose of this section is to formulate an (S, T)-version of Question 5.2. We introduce another finite set of finite primes T which satisfies the following properties.

Hypothesis 5.2.

(1) $S \cap T = \varnothing$,

(2) $\mu_{K,T} = 1$.

The analogue of Question 5.2 in the (S,T)-context is the following.

Question 5.10. *Let K/k be an abelian extension of number fields, S a finite set of places of k satisfying Hypothesis 5.1, and T a finite set of finite places satisfying Hypothesis 5.2. Fix $v \in S_{min}$ and $w \in S_L$ lying above v. With the same notation as before we ask if there exists $\eta \in (E_{L,S,T})_{1,S}$ such that*

$$\theta'_{L/k,S,T}(0) = n \cdot R_{L/k,w}(\eta). \tag{5.4}$$

We make some remarks:

(1) As in Conjecture 3.48, the uniqueness of η follows because we require $\eta \in (E_{L,S,T})_{1,S}$. If $|S| \geq 3$, we could replace this condition and require instead that $|\eta|_{w'} = 1$ for all $w' \in S_L$ not lying above v.

(2) We shall denote this question by $St(K/k, S, T, v)$, where $v \in S_{min}$. Moreover, if we want to specify the choice of $w \in S_L$ we will write $St(K/k, S, T, v, w)$. As before, the veracity of the question does not depend on the choice of w.

(3) Moreover, in the case of an affirmative answer, the $\eta \in (E_{L,S,T})_{1,S}$ satisfying

$$\theta'_{L/k,S,T}(0) = n \cdot R_{L/k,w}(\eta),$$

will be called a Stark n-unit.

(4) If S contains a split prime v then either $S_{min} = \varnothing$ or $S_{min} = \{v\}$. In the latter case, $St(K/k, S, T, v)$ is just the usual (S,T)-version of the abelian rank one Stark conjecture since $K = L$ and $n = 1$ in that case.

(5) The usual (S,T)-version of the abelian rank one of Stark conjecture for L/k, that is $St(L/k, S, T, v)$, predicts the existence of an $\varepsilon \in (E_{L,S,T})_{1,S}$, satisfying

$$\theta'_{L/k,S,T}(0) = R_{L/k,w}(\varepsilon).$$

Since $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-module when restricted to the $()_{1,S}$-subspace, one should have

$$\varepsilon = \eta^n.$$

In other words, in the case of an affirmative answer and in the setting of our question, the Stark unit for $St(L/k,S,T,v)$ should in fact be a n-th power of some other unit which we call a Stark n-unit.

(6) Another way of looking at (5.4) is to say that $R^{-1}_{L/k,w}(\theta'_{L/k,S,T}(0)) \in (E_{L,S,T})^n_{1,S}$ which is a smaller subgroup then $(E_{L,S,T})_{1,S}$.

Proposition 5.11. *Let K/k be an abelian extension of number fields and let S and T be finite sets of places satisfying Hypotheses 5.1 and 5.2. If $St(K/k,S,T,v)$ has an affirmative answer for all $v \in S_{min}$, then Conjecture 4.15 is true.*

Proof:

This is a direct consequence of Proposition 4.16

Q.E.D.

Proposition 5.12. *Let K/k be an abelian extension of number fields and let S and T be finite sets of places satisfying Hypotheses 5.1 and 5.2. If $S_{min} = \{v\}$ consists of a unique place, then $St(K/k,S,T,v)$ has an affirmative answer if and only if Conjecture 4.15 is true.*

Proof:

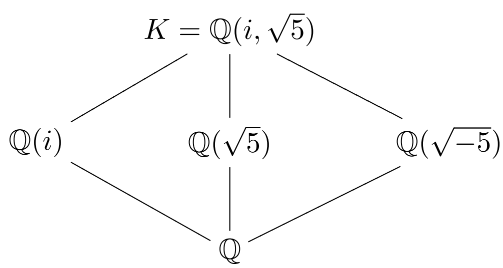
This is a consequence of Proposition 4.16 and the fact that the map

$$E_{L,S,T}/E^n_{L,S,T} \longrightarrow E_{K,S,T}/E^n_{K,S,T},$$

is injective, where $L = K^{G_v}$, and $n = |G_v|$.

Q.E.D.

We remark that it is simple to find examples where $|S_{min}| = 1$. In fact, if one enlarges S, then S_{min} stays the same or become smaller. Using this principle let us look at the biquadratic extension $K/\mathbb{Q}$ where $K = \mathbb{Q}(i,\sqrt{5})$. We have the following diagram of fields.



A simple computation shows that $S = \{\infty, 2, 5, 7\}$ is a 1-cover and $S_{min} = \{\infty, 5, 7\}$. If we toss in more primes as for instance $S = \{\infty, 2, 5, 7, 11, 13\}$ then $S_{min} = \{7\}$.

There are (S,T)-versions of Proposition 5.4 and Question 5.7.

Proposition 5.13. *Let K/k be an abelian extension of number fields, S be a finite set of places satisfying Hypothesis 5.1 and suppose that $|S| \geq 3$. Let T be another finite set of finite primes of k satisfying Hypothesis 5.2 and let $\mathfrak{p}$ be a finite prime in S_{min}. With the same notation as above, if $St(K/k, S, T, \mathfrak{p}, \mathfrak{P})$ is true with Stark n-unit η then the following statements hold:*

(1) $\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) $\mathfrak{P}^{\frac{\theta_{L/k,R,T}(0)}{n}} = (\eta)$ *as ideals in L,*

(3) $\eta \in L^\times_{\mathfrak{m}(T_L)}$,

(4) $\eta \in L^0$, *i.e. it is an anti-unit.*

Conversely, if (1) holds and if there exists $\eta \in L^\times$ for which (2),(3), and (4) hold then $St(K/k, S, T, \mathfrak{p}, \mathfrak{P})$ holds true with Stark n-unit η.

Definition 5.14. Let K/k be an abelian extension of number fields and R a finite set of places of k containing $S(K/k)$ which is a $\mathfrak{p}$-1-cover for K/k. Let also T be a finite set of finite primes satisfying $(R \cup \{\mathfrak{p}\}) \cap T = \varnothing$ and $\mu_{K,T} = 1$. Define $A_T = A_{T,R,L}$ to be the subgroup of $Cl_{L,T}$ generated by the $[\mathfrak{Q}]$ where $\mathfrak{Q}$ is a prime ideal of L lying above a prime $\mathfrak{q} \notin R \cup T$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Also, define $I^*_{L,T}$ to be the subgroup of $I_{L,T}$ (the subgroup of fractional ideals of L relatively prime with the primes in T) satisfying $I^*_{L,T}/(L^\times_{\mathfrak{m}(T_L)}) = A_T$.

Question 5.15. *With the notation as above, are the following statements true?*

(1) $\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) *For any $\mathfrak{a} \in I^*_{L,T}$, there exists $\eta_{R,T}(\mathfrak{a}) \in L^\times$ satisfying*

$$\mathfrak{a}^{\frac{\theta_{L/k,R,T}(0)}{n}} = (\eta_{R,T}(\mathfrak{a})),$$

(3) $\eta_{R,T}(\mathfrak{a}) \equiv 1 \mod {}^{\times}\mathfrak{m}(T_L)$,

(4) $\eta_{R,T}(\mathfrak{a}) \in L^0$.

We shall denote this conjecture by $BrSt(K/k, R, T)$ where R is a given $\mathfrak{p}$-1-cover.

5.3 Equivalence between the S and (S, T)-versions

Theorem 5.16. *Let K/k be an abelian extension of number fields and S a finite set of primes satisfying Hypothesis 5.1, and choose $v \in S_{min}$. As usual let $L = K^{G_v}$. If $St(K/k, S, v, w)$ has an affirmative answer then $St(K/k, S, T, v, w)$ has an affirmative answer for all finite set of finite primes T satisfying Hypothesis 5.2.*

More precisely, if η_S is a Stark n-unit for $St(K/k, S, v, w)$, then pick a w_L-th root λ of η_S. We know that $F = L(\lambda)$ is abelian over k. For each $\mathfrak{p} \in T$, fix a lift of $\left(\frac{L/k}{\mathfrak{p}}\right)$ to F and call it $\tilde{\sigma}_{\mathfrak{p}}$. Let

$$\alpha = \prod_{\mathfrak{p} \in T} \left(1 - \tilde{\sigma}_{\mathfrak{p}}^{-1} \mathbb{N}(\mathfrak{p})\right) \in \mathbb{Z}[\mathrm{Gal}(F/k)].$$

Then, $\eta_{S,T} = \lambda^{\alpha}$ is the Stark n-unit for $St(K/k, S, T, v, w)$.

Proof:

We first check that $\eta_{S,T} \in L^{\times}$. Let $\sigma \in \mathrm{Gal}(F/L)$. There exists $\zeta \in \mu_L$ such that $\lambda^{\sigma} = \zeta\lambda$. Thus

$$\eta_{S,T}^{\sigma} = (\lambda^{\alpha})^{\sigma} = (\lambda^{\sigma})^{\alpha} = (\zeta\lambda)^{\alpha} = \lambda^{\alpha} = \eta_{S,T}.$$

This last chain of equalities is true because the field F is abelian over k and $\zeta^{\alpha} = 1$. If $\pi : \mathbb{Z}[\mathrm{Gal}(F/k)] \longrightarrow \mathbb{Z}[\mathrm{Gal}(L/k)]$ denotes the usual restriction map, we have

$$\pi(\alpha) = \prod_{\mathfrak{p} \in T} \left(1 - \left(\frac{L/k}{\mathfrak{p}}\right)^{-1} \mathbb{N}(\mathfrak{p})\right).$$

Setting $\delta_T = \pi(\alpha)$, we compute

$$\begin{aligned}\theta'_{S,T}(0) &= \delta_T \theta'_S(0) \\ &= \frac{n \cdot \delta_T}{w_L} R_{L/k,w}(\eta_S)\end{aligned}$$

Further

$$\begin{aligned}\frac{n\cdot\delta_T}{w_L}R_{L/k,w}(\eta_S) &= \frac{n}{w_L}R_{L/k,w}(\eta_S^{\delta_T})\\ &= \frac{n}{w_L}R_{L/k,w}((\lambda^{\alpha})^{w_L})\\ &= n\cdot R_{L/k,w}(\eta_{S,T}).\end{aligned}$$

Remarking that $\eta_{S,T}\equiv 1 \mod {}^{\times}\mathfrak{P}$ for all $\mathfrak{P}\in T_L$, we can conclude the desired result.

Q.E.D.

Theorem 5.17. *Let K/k be an abelian extension of number fields and let S be a finite set of primes satisfying Hypothesis 5.1. Let $v\in S_{min}$, $L=K^{G_v}$, $\Gamma=\mathrm{Gal}(L/k)$, $n=[K:L]$, and Ω be a finite set of finite primes such that $S\cap\Omega=\varnothing$. Suppose that*

$$\{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})\,|\,\mathfrak{p}\in\Omega\}$$

generate $\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$ over $\mathbb{Z}$. If $St(K/k,S,T,v,w)$ has an affirmative answer for all $T=\{\mathfrak{p}\}$ where $\mathfrak{p}$ runs over all primes in Ω, then $St(K/k,S,v,w)$ has an affirmative answer.

More precisely, if $\eta_{S,\mathfrak{p}}$ is the $(S,\{\mathfrak{p}\})$-unit for $St(K/k,S,\{\mathfrak{p}\},v,w)$ and

$$w_L=\sum_{\mathfrak{p}\in\Omega}n_{\mathfrak{p}}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

then the Stark unit for $St(K/k,S,v,w)$ is given by

$$\eta_S=\prod_{\mathfrak{p}\in\Omega}\eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}}.$$

Proof:

For $\gamma\in\Gamma$, choose integers n_γ such that $\zeta^\gamma=\zeta^{n_\gamma}$ for all $\zeta\in\mu_L$. Since w_L and $\gamma-n_\gamma$ are in $\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$, one can write

$$w_L=\sum_{\mathfrak{p}\in\Omega}n_{\mathfrak{p}}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

and

$$\gamma-n_\gamma=\sum_{\mathfrak{p}\in\Omega}n_{\mathfrak{p},\gamma}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

for some $n_{\mathfrak{p}}, n_{\mathfrak{p},\gamma} \in \mathbb{Z}$. Set

$$\eta_S = \prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}},$$

and

$$\alpha_\gamma = \prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p},\gamma}}.$$

We have

$$\begin{aligned} w_L \cdot \theta'_{L/k,S}(0) &= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot (1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p}))\theta'_{L/k,S}(0) \\ &= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot \theta'_{L/k,S,\{\mathfrak{p}\}}(0) \\ &= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot n \cdot R_{L/k,w}(\eta_{S,\mathfrak{p}}) \\ &= n \cdot R_{L/k,w}\big(\prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}}\big) \\ &= n \cdot R_{L/k,w}(\eta_S). \end{aligned}$$

Noting that η_S is a S-unit having the correct absolute values, we just have to show that it satisfies the abelian condition. For that purpose, we use Theorem A.5. First, we claim that

$$\eta_S^{\gamma-n_\gamma} = \alpha_\gamma^{w_L},$$

for all $\gamma \in \Gamma$. On one hand, we have

$$\eta_S^{\gamma-n_\gamma} = \prod_{\mathfrak{p}\in\Omega}\prod_{\mathfrak{q}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}n_{\mathfrak{q},\gamma}(1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q}))},$$

and on the other hand, we have

$$\alpha_\gamma^{w_L} = \prod_{\mathfrak{p}\in\Omega}\prod_{\mathfrak{q}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p},\gamma}n_{\mathfrak{q}}(1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q}))}.$$

If we show that

$$\eta_{S,\mathfrak{p}}^{1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q})} = \eta_{S,\mathfrak{q}}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})},$$

for all $\mathfrak{p}, \mathfrak{q} \in \Omega$, then it would prove our claim. Since

$$(1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{p}))\theta'_{L/k,S,\{\mathfrak{p}\}}(0) = (1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{q}))\theta'_{L/k,S,\{\mathfrak{p}\}}(0),$$

and $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-modules when restricted to the $()_{1,S}$-component, we conclude that $\eta_{S,\mathfrak{p}}^{1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q})}$ and $\eta_{S,\mathfrak{q}}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})}$ differ by a root of unity which is congruent to 1 modulo both $\mathfrak{p}$ and $\mathfrak{q}$. Hence, this root of unity is necessarily 1 and the desired equality holds.

The second condition in Theorem A.5 is a simple computation and left to the reader. We conclude that η_S is w_L-abelian over k as desired.

Q.E.D.

5.4 Functoriality questions

In order to prove the usual functoriality properties of our question, we will use the (S, T)-version rather than the S-version since it is simpler to deal with the roots of unity in this context. If he desires, the reader will have no problem giving a direct proof of these functoriality properties in the setting of the S-version.

Proposition 5.18. *Let K/k be an abelian extension of number fields and let K' be a subextension: $k \subseteq K' \subseteq K$. Let S and T be finite sets of places satisfying Hypotheses 5.1 and 5.2. Clearly, we also have $\mu_{K',T} = 1$. Then the following statements are true:*

(1) *S is also a 1-cover for K'/k,*

(2) *One has $S_{K',min} \subseteq S_{K,min}$, where the notation should be self-explanatory,*

(3) *If $v \in S_{K',min}$ then the truth of $St(K/k, S, T, v)$ implies the truth of $St(K'/k, S, T, v)$.*

Proof:

We shall decorate all the objects of K'/k with an apostrophe as G', G'_v, etc. Let $\chi \in \widehat{G'}$ and consider $\widetilde{\chi} = \chi \circ \pi \in \widehat{G}$ where

$$\pi : G \longrightarrow G'$$

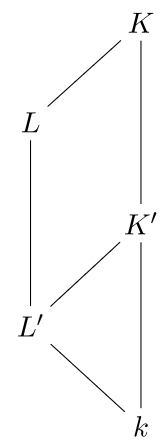
is the usual projection map. Since S is a 1-cover for K/k, there exists $v \in S$ such that $\widetilde{\chi}(G_v) = 1$. Since $\pi(G_v) = G'_v$, we get

$$\chi(G'_v) = \widetilde{\chi}(G_v) = 1,$$

that is $G'_v \subseteq \mathrm{Ker}(\chi)$. This shows that S is also a 1-cover for K'/k.

Next, if $S_{K',min} \neq \varnothing$, let $v \in S_{K',min}$. This means that there exists $\chi \in \widehat{G'}$ such that G'_v is the unique decomposition group included in $\mathrm{Ker}(\chi)$. We remark that $\widetilde{\chi}(G_v) = \chi(G'_v) = 1$. We claim that $v \in S_{K,min}$ as well. If not, there would exist another place $w \neq v$ such that $\widetilde{\chi}(G_w) = 1$, but then we would get $\chi(G'_w) = 1$, a contradiction.

As for the third part, we have the following diagram of fields.



where $L = K^{G_v}$ and $L' = K'^{G'_v}$. We remark that $L \cap K' = L'$ and therefore $n' \mid n$ where $n = |G_v| = [K : L]$ and $n' = |G'_v| = [K' : L']$. Now suppose that $St(K/k, S, T, v)$ holds true, that is there exists $\eta \in (E_{L,S,T})_{1,S}$ satisfying

$$n \cdot R_{L/k,w}(\eta) = \theta'_{L/k,S,T}(0).$$

Applying the restriction map $\mathrm{res}_{L/L'}$, we get

$$n \cdot R_{L'/k,w'}(N_{L/L'}(\eta)) = \theta'_{L'/k,S,T}(0).$$

Since $n' \mid n$, $n = n's$ for some integer s and therefore

$$n' \cdot R_{L'/k,w'}(N_{L/L'}(\eta)^s) = \theta'_{L'/k,S,T}(0).$$

In other words, $St(K'/k, S, T, v)$ is true with Stark n'-unit $N_{L/L'}(\eta)^s$.

Q.E.D.

Proposition 5.19. *Let K/k be an abelian extension of number fields and S and T be finite sets of places satisfying Hypothesis 5.1 and 5.2. Let S' be any other finite set of primes containing S such that $S' \cap T = \varnothing$, then the following statements are true.*

(1) *S' is also a 1-cover for K/k,*

(2) *One has $S'_{min} \subseteq S_{min}$,*

(3) *If $v \in S'_{min}$, then the truth of $St(K/k, S, T, v)$ implies the truth of $St(K/k, S', T, v)$.*

Proof:

The first part of the proposition is clear.

If $v \in S'_{min}$, then there exists a character $\chi \in \widehat{G}$ such that v is the unique place in S' for which $G_v \subseteq \text{Ker}(\chi)$. Since S itself is a 1-cover for K/k, we see that $v \in S$ necessarily and thus $v \in S_{min}$. This concludes the proof of the second part.

The proof of the third part is similar to the proof of Theorem 3.10.

Q.E.D.

5.5 Some results

The following theorem is essentially due to Erickson (see [19]).

Theorem 5.20. *Let K/k be a finite abelian extension of number fields and let S be a finite set of places satisfying Hypotheses 5.1. If $v \in S_{min}$, suppose that G_v is cyclic generated by an element $\sigma(v)$. Suppose also that there exists a subset $S' \subseteq S$ containing only unramified finite primes and v and such that S' is a 1-cover for K/k. Let $S_0 = S \setminus S'$ and set $S_v = S_0 \cup \{v\}$. If the classical abelian rank one Stark conjecture $St(L/k, S_v, v)$ is true, then so is $St(K/k, S, v)$ ($L = K^{G_v}$ as usual).*

Proof:

Let η be a Stark unit for $St(L/k, S_v, v)$. By Theorem 5.19, a Stark unit for $St(L/k, S, v)$ is given by

$$\varepsilon = \eta^{\alpha},$$

where

$$\alpha = \prod_{\substack{\mathfrak{p} \in S' \\ \mathfrak{p} \neq v}} (1 - \sigma_{\mathfrak{p}}^{-1}) \in \mathbb{Z}[\Gamma].$$

Consider

$$\tilde{\alpha} = \prod_{\substack{\mathfrak{p} \in S' \\ \mathfrak{p} \neq v}} (1 - \tilde{\sigma}_{\mathfrak{p}}^{-1}) \in \mathbb{Z}[G],$$

where $\tilde{\sigma}$ denotes a lift of σ to G. Moreover, let res : $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[\Gamma]$ be the usual restriction map. Since

$$\left(\frac{K/k}{\mathfrak{p}}\right)\Big|_L = \left(\frac{L/k}{\mathfrak{p}}\right),$$

we have $\mathrm{res}(\tilde{\alpha}) = \alpha$. Therefore, we also have $\varepsilon = \eta^{\tilde{\alpha}}$. If we show that $n = |G_v|$ divides α in $\mathbb{Z}[\Gamma]$ then we would be done. This fact follows from the two following lemmata. Indeed, Lemma 5.22 below implies that $\tilde{\alpha} \in N\sigma(v) \cdot \mathbb{Z}[G]$, where

$$N\sigma(v) = 1 + \sigma(v) + \ldots + \sigma(v)^{n-1}.$$

Hence, $\alpha = \mathrm{res}(\tilde{\alpha}) \in n \cdot \mathbb{Z}[\Gamma]$.

Q.E.D.

Lemma 5.21. *Let G be a finite abelian group and let S be a subset of G satisfying the following condition: For all $\chi \in \widehat{G}$, $\chi \neq \chi_1$, there exists $\sigma \in S$ such that $\chi(\sigma) = 1$. Then*

$$\prod_{\sigma \in S} (1 - \sigma) = 0 \in \mathbb{Z}[G].$$

Proof:

By applying e_χ one χ at the time, this identity should be clear.

Q.E.D.

Lemma 5.22. *Let G and S be as in the previous lemma. Choose any $\sigma_0 \in S$, then*

$$\prod_{\substack{\sigma \in S \\ \sigma \neq \sigma_0}} (1 - \sigma) \in N\sigma_0 \cdot \mathbb{Z}[G],$$

where $N\sigma_0 = 1 + \sigma_0 + \ldots + \sigma_0^{\mathrm{ord}(\sigma_0)-1}$.

Proof:

Let $H =< \sigma_0 >$. This identity should be clear in view of the fact that for any group G and subgroup $H \leq G$ one has $\mathbb{Z}[G]^H = N_H \cdot \mathbb{Z}[G]$, where $N_H = \sum_{h \in H} h$.

Q.E.D.

The next theorem is also adapted from [19].

Theorem 5.23. *Let K/k be a finite abelian extension of number fields, and let S be a finite set of places satisfying Hypothesis 5.1. Suppose that there exists a subset $S' \subseteq S$ which consists of only unramified finite primes and one infinite real place and such that S' is a 1-cover for K/k. Then, the following hold:*

(1) *If v is the unique infinite place in S_{min}, then $St(K^{G_v}/k, \{v\} \cup (S \setminus S'), v)$ implies the truth of $St(K/k, S, v)$.*

(2) *If $\mathfrak{p} \in S_{min}$ is a finite unramified prime, then $St(K^{G_\mathfrak{p}}/k, \{v, \mathfrak{p}\} \cup (S \setminus S'), \mathfrak{p})$ implies the truth of $St(K/k, S, \mathfrak{p})$.*

Proof:

The first point (when v is infinite) follows from Theorem 5.20. If $\mathfrak{p} \in S_{min}$ is a finite unramified prime, let η be a Stark unit for $St(K^{G_\mathfrak{p}}/k, \{v, \mathfrak{p}\} \cup (S \setminus S'), \mathfrak{p})$. We know that a Stark unit ε for $St(K^{G_\mathfrak{p}}/k, S, \mathfrak{p})$ is given by $\varepsilon = \eta^\alpha$, where

$$\alpha = \prod_{\substack{\mathfrak{q} \in S' \\ \mathfrak{q} \neq v, \mathfrak{p}}} (1 - \sigma_\mathfrak{q}^{-1}) \in \mathbb{Z}[\Gamma],$$

and where we view the Frobenius automorphisms as being in Γ. Remark also that $\tau \notin G_\mathfrak{p}$ otherwise $\mathfrak{p}$ would not be in S_{min}. We shall denote the coset $\tau G_\mathfrak{p}$ simply by τ as well. Lemma 5.22 implies that

$$(1 - \tau) \cdot \prod_{\substack{\mathfrak{q} \in S' \\ \mathfrak{q} \neq v, \mathfrak{p}}} (1 - \sigma_\mathfrak{q}^{-1}) \in |G_\mathfrak{p}| \cdot \mathbb{Z}[\Gamma].$$

Write

$$\prod_{\substack{\mathfrak{q} \in S' \\ \mathfrak{q} \neq v, \mathfrak{p}}} (1 - \sigma_\mathfrak{q}^{-1}) = \sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma,$$

for some $a_\gamma \in \mathbb{Z}$. Let $\gamma_1, \ldots, \gamma_s$ be coset representatives for $\Gamma / < \tau >$, then

$$\begin{aligned}
\sum_{\gamma \in \Gamma} a_\gamma \gamma &= \sum_{i=1}^{s} a_{\gamma_i} \gamma_i + \sum_{i=1}^{s} a_{\gamma_i \tau} \gamma_i \tau \\
&= \sum_{i=1}^{s} a_{\gamma_i \tau} (\gamma_i + \gamma_i \tau) + \sum_{i=1}^{s} (a_{\gamma_i} - a_{\gamma_i \tau}) \gamma_i \\
&= (1 + \tau) \sum_{i=1}^{s} a_{\gamma_i \tau} \gamma_i + \sum_{i=1}^{s} (a_{\gamma_i} - a_{\gamma_i \tau}) \gamma_i.
\end{aligned}$$

Since $\varepsilon^{1+\tau} = 1$, we get

$$\varepsilon^{\alpha} = \varepsilon^{\sum_{i=1}^{s}(a_{\gamma_i} - a_{\gamma_i\tau})\gamma_i}. \tag{5.5}$$

Now, we have

$$\begin{aligned}(1-\tau)\cdot\alpha &= (1-\tau^2)\sum_{i=1}^{s} a_{\gamma_i\tau}\gamma_i + (1-\tau)\sum_{i=1}^{s}(a_{\gamma_i} - a_{\gamma_i\tau})\gamma_i \\ &= \sum_{i=1}^{s}(a_{\gamma_i} - a_{\gamma_i\tau})\gamma_i - \sum_{i=1}^{s}(a_{\gamma_i} - a_{\gamma_i\tau})\gamma_i\tau \\ &= \sum_{\gamma\in\Gamma}(a_\gamma - a_{\gamma\tau})\gamma.\end{aligned}$$

This implies that $|G_{\mathfrak{p}}| \,|\, (a_\gamma - a_{\gamma\tau})$ for all $\gamma \in \Gamma$ and equation (5.5) shows that ε^{α} is a $|G_{\mathfrak{p}}|$-th power in $K^{G_{\mathfrak{p}}}$. This is what we wanted to show.

Q.E.D.

Proposition 5.24. *Let K/k be an abelian extension of number fields. Let S be a finite set of places satisfying Hypothesis 5.1 and suppose $|S| \geq 3$. Let $\mathfrak{p}$ be a finite prime in S_{min} and set $R = S \setminus \{\mathfrak{p}\}$. For each $\gamma \in \Gamma$, let n_γ be an integer satisfying $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_L$. Suppose that the following two conditions hold:*

(1) $\frac{w_L\theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) $\frac{(\gamma - n_\gamma)\theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$ *for all* $\gamma \in \Gamma$.

If $\mathfrak{P}$ is a principal prime ideal of L lying above $\mathfrak{p}$, then $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ holds true.

Proof:

We use Proposition 5.4. Since $\mathfrak{P}$ is principal, there exists $u \in L^\times$ such that $\mathfrak{P} = (u)$. Therefore

$$\mathfrak{P}^{\frac{w_L\theta_{L/k,R}(0)}{n}} = (u^{\frac{w_L\theta_{L/k,R}(0)}{n}}).$$

Now $u^{\frac{w_L\theta_{L/k,R}(0)}{n}}$ is an anti-unit and also w_L-abelian over k by Lemmata 3.21 and 3.22.

Q.E.D.

The following theorem takes care of the simplest possible case, namely the case of biquadratic extensions.

Theorem 5.25. *Let K/k be a biquadratic extension and let S be a finite set of primes of k satisfying Hypothesis 5.1. Then $St(K/k, S, v)$ has an affirmative answer for any $v \in S_{min}$.*

Proof:

If S contains a split prime, we are in the usual setting of the classical abelian rank one Stark conjecture and the result is known due to Sand (see [44] and [45]). Suppose that S does not contain a split prime. Using the fact that $S \neq S_{min}$, we claim that $|S| \geq 4$. Indeed, by Theorem 4.28, there are 3 minimal cocyclic subgroups for K/k, namely the three subgroups of order two. Since S is a 1-cover, we necessarily have $|S| \geq 3$ by Theorem 4.19. Since $S \neq S_{min}$, we also get that $|S| \geq 4$ by Theorem 4.20. We can now conclude the desired result by Theorem 3.11.

Q.E.D.

5.5.1 The place $v \in S_{min}$ is real infinite

The next theorem is motivated by [15].

Theorem 5.26. *Let K/k be an abelian extension of number fields with Galois group G and let S be a finite set of places satisfying Hypothesis 5.1. Suppose also that $|S| \geq 3$. Let $v \in S_{min}$ be an infinite real place such that v does not split all the way up to K (otherwise, we would be in the case of the usual abelian rank one Stark conjecture). Fix a place w of $L = K^{G_v}$ lying above v. Assuming the Gross conjecture, there exists a unit $\eta \in E_{L,S}$ satisfying*

$$\theta'_{L/k,S}(0) = \frac{2}{w_L} R_{L/k,w}(\eta),$$

and such that $|\eta|_{w'} = 1$ for all places $w' \in S_L$ not lying above v.

Proof:

Since v is a real infinite place, the local reciprocity map

$$\mathrm{rec}_w : L_w^\times \longrightarrow \mathrm{Gal}(K/L),$$

is just the map $x \mapsto \mathrm{sgn}_w(x)$ which is 1 if and only if x is positive at w. Let T be any set satisfying Hypothesis 5.2, since S is a 1-cover for K/k, the Gross conjecture implies that

$$\mathrm{rec}_w(\varepsilon_{S,T}^{\gamma}) = 1,$$

for all $\gamma \in \Gamma$. In other words, $\varepsilon_{S,T}^{\gamma}$ is positive at w for all $\gamma \in \Gamma$.

Let $\varepsilon \in L^{\times}$ be a Stark unit for the usual abelian rank one Stark conjecture $St(L/k, S, v)$. We can suppose that ε is positive at w by multiplying by -1 if necessary. Take any prime $\mathfrak{p}$ of k which splits completely in L/k and is relatively prime to S and w_K. Then $T = \{\mathfrak{p}\}$ satisfies Hypothesis 5.2 by Proposition 3.43. Let also $\mathfrak{P}$ be any prime of L lying above $\mathfrak{p}$ and let $\sigma_{\mathfrak{p}} = \left(\frac{L(\sqrt{\varepsilon})/k}{\mathfrak{p}}\right)$ be the Frobenius automorphism. By Theorem 5.16, we have

$$\varepsilon_{S,T} = \sqrt{\varepsilon}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})} = \left(\sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-\mathbb{N}(\mathfrak{p})}\right)^{\sigma_{\mathfrak{p}}^{-1}}.$$

Now

$$\begin{aligned} \sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-\mathbb{N}(\mathfrak{p})} &= \sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-1+1-\mathbb{N}(\mathfrak{p})} \\ &= \left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} \varepsilon^{\frac{1-\mathbb{N}(\mathfrak{p})}{2}}. \end{aligned}$$

This shows that

$$\varepsilon_{S,T} = \left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} \varepsilon^{\frac{1-\mathbb{N}(\mathfrak{p})}{2}}.$$

Since both $\varepsilon_{S,T}$ and ε are positive at w, we get

$$\left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} = 1.$$

By Hasse's principle for powers (Theorem 2.15), we conclude that ε is a square in L, that is $\varepsilon = \eta^2$ for some $\eta \in L^{\times}$. Since ε has the correct absolute values, we have $|\eta|_{w'} = 1$ for all places $w' \in S_L$ not lying above v. Moreover

$$\theta'_{L/k,S}(0) = \frac{1}{w_L} R_{L/k,w}(\varepsilon) = \frac{2}{w_L} R_{L/k,w}(\eta).$$

This is what we wanted to show.

Q.E.D.

In this last proof, we used the Gross conjecture for a real infinite place. The Gross conjecture in this setting is known to be true when the top field is a CM-field, see [41].

Remark 18. *Because of this last theorem, only the abelian condition is left to investigate in the case where $v \in S_{min}$ is a real infinite place (assuming the usual abelian rank one Stark conjecture and the Gross conjecture). So far, we have not found any general* *strategy to attack this problem. In all cases verified numerically, the abelian condition is satisfied.*

5.5.2 The place $\mathfrak{p} \in S_{min}$ is finite unramified

Theorem 5.27 below was proved in the context of Conjecture 4.4 by Erickson in [19].

Theorem 5.27. *Let K/k be a finite abelian extension of number fields, and let S be a finite set of places satisfying Hypothesis 5.1. Suppose that there exists a subset $S' \subseteq S$ which consists of only unramified finite primes and such that S' is a 1-cover for K/k. Let $S_0 = S \setminus S'$. For all $\mathfrak{p} \in S_{min}$, set $S_{\mathfrak{p}} = S_0 \cup \{\mathfrak{p}\}$. If the usual abelian rank one Stark conjecture $St(K^{G_{\mathfrak{p}}}/k, S_{\mathfrak{p}}, \mathfrak{p})$ is true for all $\mathfrak{p} \in S_{min}$, then $St(K/k, S, \mathfrak{p})$ has an affirmative answer for all $\mathfrak{p} \in S_{min}$.*

Proof:

Immediate from Theorem 5.20.

Q.E.D.

We remark that if this last theorem applies, then S_{min} consists of only finite unramified primes and $S_{min} \subseteq S'$.

Theorem 5.28. *Let K/k be an abelian extension of number fields and R be a $\mathfrak{p}$-1-cover satisfying $R \supseteq S(K/k)$. For each $\gamma \in \Gamma$, let n_γ be an integer satisfying $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_L$. Suppose that*

(1) $\frac{w_L \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) $\frac{(\gamma - n_\gamma)\theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$ *for all* $\gamma \in \Gamma$.

If $h_L = 1$, then $BrSt(K/k, R)$ has an affirmative answer.

Proof:

Immediate from Proposition 5.24.

Q.E.D.

Theorem 5.29. *Let K/k be an abelian extension of number fields with Galois group G and let R be a $\mathfrak{p}$-1-cover for K/k. Let T be a finite set of primes satisfying*

(1) $\mu_{K,T} = 1$,

(2) $(R \cup \{\mathfrak{p}\}) \cap T = \varnothing$.

As above, set $L = K^{G_\mathfrak{p}}$. *Then Gross's conjecture for the data* $(K/L/k, S, T)$ *implies that*

$$\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma],$$

where we recall $\Gamma = \mathrm{Gal}(L/k)$ *and* $n = |G_\mathfrak{p}|$.

Proof:

Let $S = R \cup \{\mathfrak{p}\}$ and let $\mathfrak{P}$ be a prime ideal of L lying above $\mathfrak{p}$. Since $\mathfrak{P}$ is unramified in K/L the local reciprocity map is simple to described: One has $rec_\mathfrak{P}(x) = \sigma_\mathfrak{p}^n$, where $n = \mathrm{ord}_\mathfrak{P}(x)$ and $\sigma_\mathfrak{p}$ is the Frobenius automorphism at $\mathfrak{p}$. We remark that

$$\sigma_\mathfrak{p} = \left(\frac{K/k}{\mathfrak{p}}\right) = \left(\frac{K/L}{\mathfrak{P}}\right).$$

Suppose that $St(L/k, S, T, \mathfrak{p}, \mathfrak{P})$ is true and let $\varepsilon_{S,T}$ be the corresponding Stark unit. Gross's conjecture $\mathrm{Gr}(K/L/k, S, T)$ implies that

$$rec_\mathfrak{P}(\varepsilon_{S,T}^{\gamma^{-1}}) = \prod_{\substack{\sigma \in G \\ \sigma|_L = \gamma}} \sigma^{\zeta_{K/k,S,T}(0,\sigma^{-1})},$$

for all $\gamma \in \Gamma$. Since S is a 1-cover for K/k, we have

$$\theta_{K/k,S,T}(0) = 0,$$

and therefore $\zeta_{K/k,S,T}(0, \sigma) = 0$, for all $\sigma \in G$. Hence Gross's conjecture predicts that

$$rec_\mathfrak{P}(\varepsilon_{S,T}^{\gamma}) = 1, \tag{5.6}$$

for all $\gamma \in \Gamma$. On the other hand, we have

$$\mathrm{ord}_\mathfrak{P}(\varepsilon_{S,T}^{\gamma}) = \zeta_{L/k,R,T}(0, \gamma),$$

for all $\gamma \in \Gamma$. Thus

$$rec_\mathfrak{P}(\varepsilon_{S,T}^{\gamma}) = \sigma_\mathfrak{p}^{\zeta_{L/k,R,T}(0,\gamma)}, \tag{5.7}$$

for all $\gamma \in \Gamma$. Combining equations (5.6) and (5.7), we get

$$f_{\mathfrak{p}} \,|\, \zeta_{L/k,R,T}(0, \gamma),$$

for all $\gamma \in \Gamma$, where $f_{\mathfrak{p}}$ is the order of $\sigma_{\mathfrak{p}}$ in G. Note that since $\mathfrak{p}$ is unramified, $f_{\mathfrak{p}} = |G_{\mathfrak{p}}|$. At last, the desired result follows from

$$\theta_{L/k,R,T}(0) = \sum_{\gamma \in \Gamma} \zeta_{L/k,R,T}(0, \gamma) \cdot \gamma^{-1}.$$

Q.E.D.

Remark 19. *This last result takes care of point* (1) *of* $BrSt(K/k, R, T)$ *(Question 5.15) assuming Gross's conjecture. Combined with Lemma 3.50, it takes care of point* (1) *of* $BrSt(K/k, R)$ *(Question 5.7) as well.*

5.6 Further results when the base field is $\mathbb{Q}$

In this section, we assume that the base field is $\mathbb{Q}$. We shall use the reciprocity law presented in Section 3.7 and Hasse's principle for powers explained in Section 2.1.2 in order to give further evidence to a positive answer to our question when $p \in S_{min}$ is a finite unramified prime.

Proposition 5.30. *Let* $K/\mathbb{Q}$ *be an abelian extension with Galois group* G *and let* R *be a* p-1*-cover containing* $S(K/\mathbb{Q})$*. Let also* $L = K^{G_p}$*,* $n = |G_p|$ *and set* $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$*. Let* $\mathfrak{p}$ *be a prime ideal of* L *lying above* p *and let* $\varepsilon_{L,R}(\mathfrak{p})$ *be the Brumer-Stark element at* $\mathfrak{p}$ *given explicitly in terms of a product of Gauss sums as explained in Section 3.6. If* l *is a prime number, set* $a_l = \min(\mathrm{ord}_l(n), \mathrm{ord}_l(w_L))$ *so that* l^{a_l} *is the biggest power of* l *which divide both* n *and* w_L*. Then, for any* $\gamma \in \Gamma$*, we have*

$$\left(\frac{\varepsilon_{L,R}(\mathfrak{p}^{\gamma})}{\mathfrak{q}}\right)_{L,l^{a_l}} = 1,$$

for all but finitely many prime ideals $\mathfrak{q}$ *of* L*.*

Proof:

Let us write

$$w_L \cdot \theta_{L/\mathbb{Q},R}(0) = \sum_{\sigma \in \Gamma} a_R(\gamma)\gamma^{-1}.$$

We know that

$$\frac{w_L \cdot \theta_{L/\mathbb{Q},R}(0)}{n} \in \mathbb{Z}[\Gamma],$$

by Remark 19. This means that $l^{a_l} \,|\, n \,|\, a_R(\gamma)$ for all $\gamma \in \Gamma$. Given a prime ideal $\mathfrak{q}$ of L which is relatively prime with the conductor of $L/\mathbb{Q}$ and the prime p where $\mathfrak{p} \,|\, p$, we have

$$\begin{aligned}\left(\frac{\varepsilon_{L,R}(\mathfrak{p}^\gamma)}{\mathfrak{q}}\right)_{L,l^{a_l}} &= \left(\frac{\varepsilon_{L,R}(\mathfrak{p}^\gamma)}{\mathfrak{q}}\right)_{L,w_L}^{\frac{w_L}{l^{a_l}}} \\ &= \left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{L,w_L}^{\frac{w_L \cdot a_R(\gamma)}{l^{a_l}}},\end{aligned}$$

by Proposition 2.11 and the reciprocity law (Proposition 3.30). However, $a_R(\gamma)/l^{a_l} \in \mathbb{Z}$ by what we said at the beginning of the proof. Therefore,

$$\left(\frac{\mathbb{N}(\mathfrak{q})}{\mathfrak{p}}\right)_{L,w_L}^{\frac{w_L \cdot a_R(\gamma)}{l^{a_l}}} = 1.$$

Q.E.D.

Corollary 5.31. *With the same notation as in the previous proposition, if l is an odd prime, then $\varepsilon_{L,R}(\mathfrak{p}^\gamma)$ is an l^{a_l}-th power in L, for all $\gamma \in \Gamma$.*

Proof:

This follows form Hasse's principle for powers (Theorem 2.15).

Q.E.D.

Remark 20. *In some special cases, this last corollary takes care of point (2) and (3) of $BrSt(K/k, R)$ (Question 5.7). For instance, look at Example B.2.4. This last Corollary shows that the corresponding units for K_{2617}, K_7, and K_{13} are 3rd powers of other units which are automatically anti-units.*

Chapter 6

Conclusion

The goal of this thesis was to study further a generalization of the abelian rank one Stark conjecture when *there is not necessarily a split prime* in the set S. Such a conjecture was formulated for the first time in [19] and is referred to as *the extended abelian rank one Stark Conjecture*. The (S,T)-version of the extended abelian rank one Stark conjecture is contained in [17] where a formulation of an arbitrary order of vanishing situation is given.

Instead of studying the original conjecture, we formulated a *stronger question* (Question 5.2) which seems *easier to investigate both numerically and theoretically.*

This allowed us to deduce some consequences, most notably perhaps a *potential generalization of the Brumer-Stark conjecture* predicting, among other things, an integrality property of the Stickelberger element $w_K\theta_{K/k,S}(0)$ (Point (1) of Question 5.7). Assuming a *conjecture of Gross*, we showed that this property is in fact satisfied (Theorem 5.29). We might point out here, that this theorem was our motivation to introduce the (S,T)-version of our question (Question 5.10) since as far as we know, Gross's conjecture has only been formulated in this context. The (S,T)-version also has obvious advantages over the S-version, but we think that both formulations should be studied on equal footing. It would be interesting to see if point (1) of Proposition 5.4 holds true as well. This corresponds to the case where $\mathfrak{p} \in S_{min}$ is finite and ramified and we have not been able to show the corresponding integrality property in this case. One difficulty is that the local reciprocity map is harder to describe for ramified primes than for unramified ones. The numerical computations presented in Appendix B suggest that it might be true.

When the prime $v \in S_{min}$ of Question 5.7 is infinite real, an argument of

Dummit and Hayes adapted to our purpose shows the *existence of a unit (assuming the abelian rank one Stark conjecture and the Gross conjecture) satisfying all properties of Question 5.2 except the abelian condition* (Theorem 5.26). We have not found any general strategy to attack the abelian condition which seems to be the most mysterious prediction of the abelian rank one Stark conjecture. Dummit and Hayes's argument uses Hasse's principle for powers by showing that the *positive Stark unit* is locally a square at $\mathfrak{P}$, where $\mathfrak{P}$ runs over a set of primes of density one. A Stark unit is only defined up to a root of unity and in the case where the split prime is real infinite, there are only two roots of unity. By requiring a Stark unit to be positive, one removes any ambiguity. On the other hand, if the split prime is finite, there is no simple way to specify one particular Stark unit. If one intends to show Question 5.2 by proving that certain Stark units are powers, the *determination of the argument of a Stark unit* becomes important. In any case, it seems to us that such an approach would not include the abelian condition.

We introduced a simple group theoretical notion: The *minimal cocyclic subgroups*. As explained in Section 4.4, those are related with the notion of 1-cover and give a handy way of thinking about them. Using those minimal cocyclic subgroups, we made some numerical computations when the base field is $\mathbb{Q}$ and all these satisfy Question 5.2 and thus also the extended abelian rank one Stark conjecture *providing more evidence* in support of a positive answer. Indeed, the extended abelian rank one Stark conjecture or Question 5.2 are still unknown in this simplest case when the base field is $\mathbb{Q}$. One could wonder if the explicit description of Stark units in terms of Gauss sums or cyclotomic units is of any use here since it provided the solution of the classical abelian rank one Stark conjecture when the base field is $\mathbb{Q}$. Would it be possible to find a criterion in order to tell whether or not a given product of Gauss sums is a certain power in a given algebraic number field?

As far as the numerical computations are concerned, we just remark that in all cases where we tested numerically the potential generalization of the Brumer-Stark conjecture (Question 5.7), the Stickelberger element

$$\frac{w_L \cdot \theta_{L/\mathbb{Q},R}(0)}{n}$$

annihilated the whole class group Cl_L (and the anti-unit condition was also satisfied). Thus, we did not even bother to try to compute the subgroup $A_{R,L}$ of Section 5.1.2.

There is a plethora of further directions to be investigated and we will only mention two of them which are of special interest to us. Firstly, it would be highly interesting to work out what the Equivariant Tamagawa Number Conjecture of Bloch and Kato says concretely ("à la Stark") in the case where S is a 1-cover not containing any split prime. Secondly, given a finite abelian extension of number fields L/k, a finite set of primes R containing $S(L/k)$ and an integer n, are there conditions which guarantee the existence of a prime $\mathfrak{p}$ of k and of an abelian extension K/k such that $L \subseteq K$, $G_{\mathfrak{p}} = \mathrm{Gal}(K/L)$, $n = |G_{\mathfrak{p}}|$, $S = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k and $\mathfrak{p} \in S_{min}$? If yes, this would give information on the greatest common divisor of the coefficients of $w_L \cdot \theta_{L/k,S}(0)$. It might be interesting to compare this with the work of Hayes and Smith [48].

Appendix A

Some Kummer theory

We restrict ourselves to the characteristic 0 case. Kummer theory describes the abelian extensions of a number field K whose Galois group has exponent n a divisor of w_K. Let n be such an integer. There is a bijection between the subgroups $K^\times/(K^\times)^n$ and the abelian extensions of K of exponent n. This correspondence goes as follows: To a subgroup $\Delta/(K^\times)^n$ of $K^\times/(K^\times)^n$ corresponds the extension

$$K(\sqrt[n]{\Delta}) = K(\sqrt[n]{\alpha} \,|\, \alpha \in \Delta).$$

The cyclic extensions are of the form $K(\sqrt[n]{\alpha})$ for some $\alpha \in K^\times$. We refer to [1] for this correspondence.

In general (if n is not a prime number), it is hard to give a precise description of the ramification in a Kummer extension. The only thing we need to know in this thesis (for the definition of the m-th power residue symbol) is the following proposition.

Proposition A.1. *Let K be a number field, n an integer such that $n \,|\, w_K$, and $\alpha \in K^\times$. If $\mathfrak{p}$ is a prime of K which is relatively prime to n and α, then the extension $K(\sqrt[n]{\alpha})/K$ is unramified at $\mathfrak{p}$.*

Proof:

We refer to [1] Theorem 4 page 23 for a proof of this fact.

Q.E.D.

If K is a Galois extension of k with Galois group G then $K^\times/(K^\times)^n$ is acted upon by G and it is possible to formulate Kummer theory taking into account the action of this Galois group. See [20] for this equivariant Kummer theory as one might call it.

Given an abelian extension K/k of number fields, the extensions of the form $K(\varepsilon^{1/w_K})/k$ appear recurrently in relation with the abelian rank one Stark conjecture. There are criteria which tell us when the Kummer extension $K(\varepsilon^{1/w_K})$ of K is Galois, central or abelian over k and this is what we would like to recall in this Appendix.

We start with the following simple lemma.

Lemma A.2. *Let C be a Γ-module and suppose that C is a cyclic group. Then there exist integers n_γ such that $c^\gamma = c^{n_\gamma}$ for all $c \in C$ (We think of C as being a multiplicative group).*

Proof:

Suppose that $C =< c >$, and let n_γ be such that $c^\gamma = c^{n_\gamma}$. Given any integer $i \in \mathbb{Z}$, we have $(c^i)^\gamma = (c^\gamma)^i = (c^i)^{n_\gamma}$.

Q.E.D.

Theorem A.3. *Suppose that K/k is a Galois extension of fields of characteristic 0 with Galois group Γ and suppose that $\mu_m \subseteq K^\times$. Let α be such that $\alpha^m = a \in K^\times$. The extension $K(\alpha)/k$ is Galois if and only if there exist integers n_γ for $\gamma \in \Gamma$ such that*

$$a^{\gamma - n_\gamma} \in (K^\times)^m,$$

for all $\gamma \in \Gamma$.

Proof:

Suppose first that we are given integers n_γ satisfying the condition of the theorem. We have to show that $K(\alpha)/k$ is Galois. Let $\gamma \in \Gamma$ and pick any lift $\tilde{\gamma} \in \mathrm{Hom}_k(K(\alpha), \overline{k})$. We have

$$\left(\alpha^{\tilde{\gamma} - n_\gamma}\right)^m = a^{\gamma - n_{n_\gamma}} = a_\gamma^m,$$

for some $a_\gamma \in K^\times$ by hypothesis. Therefore, there exists $\zeta \in \mu_m$ such that

$$\frac{\alpha^{\tilde{\gamma} - n_\gamma}}{a_\gamma} = \zeta,$$

and thus $\alpha^{\tilde{\gamma}} = \zeta \cdot a_\gamma \cdot \alpha^{n_\gamma} \in K(\alpha)$. We conclude that $K(\alpha)/k$ is Galois.

Conversely, suppose that $K(\alpha)/k$ is Galois, and let $G = \mathrm{Gal}(K(\alpha)/K)$. The group G is a cyclic group, a fortiori an abelian group, and therefore Γ acts on G by inner automorphisms. More precisely, let s be a set theoretic section of the restriction map $\mathrm{Gal}(K(\alpha)/k) \longrightarrow \Gamma$, then the action of Γ on G is given by

$$\gamma \cdot \sigma = s(\gamma) \cdot \sigma \cdot s(\gamma)^{-1}.$$

By Lemma A.2, there exist integers t_γ, m_γ such that

$$\zeta^\gamma = \zeta^{t_\gamma},$$

for all $\zeta \in \mu_m$ and also

$$\gamma \cdot \sigma = \sigma^{m_\gamma},$$

for all $\sigma \in G$. For $\gamma \in \Gamma$, let $n_\gamma = m_{\gamma^{-1}} t_\gamma \in \mathbb{Z}$. We claim that

$$\alpha^{s(\gamma)-n_\gamma} \in K^\times.$$

Indeed, note first that if $\sigma \in G$ then there exists $\zeta_\sigma \in \mu_m$ such that $\alpha^\sigma = \zeta_\sigma \alpha$. Now, let $\sigma \in G$, then

$$\begin{aligned}
\sigma\left(\frac{s(\gamma)\alpha}{\alpha^{n_\gamma}}\right) &= \frac{\sigma s(\gamma)\alpha}{\sigma(\alpha)^{n_\gamma}} \\
&= \frac{s(\gamma)s(\gamma)^{-1}\sigma s(\gamma)\alpha}{\zeta_\sigma^{n_\gamma}\alpha^{n_\gamma}} \\
&= \frac{s(\gamma)(\gamma\cdot\sigma)\alpha}{\zeta_\sigma^{n_\gamma}\alpha^{n_\gamma}} \\
&= \frac{s(\gamma)\sigma^{m_{\gamma^{-1}}}\alpha}{\zeta_\sigma^{n_\gamma}\alpha^{n_\gamma}} \\
&= \frac{s(\gamma)\zeta_\sigma^{m_{\gamma^{-1}}}\alpha}{\zeta_\sigma^{n_\gamma}\alpha^{n_\gamma}} \\
&= \frac{\zeta_\sigma^{n_\gamma}s(\gamma)\alpha}{\zeta_\sigma^{n_\gamma}\alpha^{n_\gamma}} \\
&= \frac{s(\gamma)\alpha}{\alpha^{n_\gamma}}.
\end{aligned}$$

Therefore, $\alpha^{s(\gamma)-n_\gamma} \in K^\times$ and

$$\left(\alpha^{s(\gamma)-n_\gamma}\right)^m = a^{\gamma-n_\gamma}.$$

This is what we wanted to show.

Q.E.D.

Theorem A.4. *With the same notation as in the previous theorem, the extension $K(\alpha)/k$ is central if and only if given integers n_γ satisfying $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_m$, one has*

$$a^{\gamma - n_\gamma} \in (K^\times)^m,$$

for all $\gamma \in \Gamma$

Proof:

Suppose that $K(\alpha)/k$ is a central extension and let n_γ be integers satisfying $\zeta^\gamma = \zeta^{n_\gamma}$. Using the same notation as in the second part of the proof of the last lemma, we see that the action of Γ on G is trivial since $K(\alpha)/k$ is central. Therefore, one can take $m_\gamma = 1$ for all $\gamma \in \Gamma$ and we conclude the desired result.

Conversely, remark first that the extension is Galois by the last lemma. Let n_γ be given, and write $a^{\gamma - n_\gamma} = a_\gamma^m$ for some $a_\gamma \in K^\times$. Let $\tilde{\gamma}$ be any lift of γ to $\mathrm{Gal}(K(\alpha)/k)$. As in the previous proof, we know that there exists $\zeta \in \mu_m$ such that

$$\alpha^{\tilde{\gamma}} = \zeta a_\gamma \alpha^{n_\gamma}.$$

Thus, given $\sigma \in G$, one has

$$(\alpha^{\tilde{\gamma}})^\sigma = \zeta a_\gamma \alpha^{\sigma n_\gamma} = \zeta a_\gamma \alpha^{\sigma n_\gamma}. \tag{A.1}$$

We also know that there exits $\zeta_\sigma \in \mu_m$ such that $\alpha^\sigma = \zeta_\sigma \alpha$. We then have

$$(\alpha^\sigma)^{\tilde{\gamma}} = \zeta_\sigma^{\tilde{\gamma}} \alpha^{\tilde{\gamma}} = \zeta_\sigma^{\tilde{\gamma}} \zeta a_\gamma \alpha^{n_\gamma}, \tag{A.2}$$

but

$$\zeta \alpha^{\sigma n_\gamma} = \zeta \zeta_\sigma^{n_\gamma} \alpha^{n_\gamma},$$

and thus we conclude that equations (A.1) and (A.2) are equal. This shows that G is in the center of $\mathrm{Gal}(K(\alpha)/k)$.

Q.E.D.

Theorem A.5. *With the same notation as in the previous theorem, suppose moreover that K/k is abelian. The extension $K(\alpha)/k$ is abelian if and only if given integers n_γ*

satisfying $\zeta^\gamma = \zeta^{n_\gamma}$ *for all* $\zeta \in \mu_m$ *there exist* $a_\gamma \in K^\times$ *such that*

$$a^{\gamma - n_\gamma} = a_\gamma^m,$$

for all $\gamma \in \Gamma$ *and such that*

$$a_{\gamma_1}^{\gamma_2 - n_{\gamma_2}} = a_{\gamma_2}^{\gamma_1 - n_{\gamma_1}},$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

Proof:

Suppose first that the two conditions are satisfied, we want to show that the extension $K(\alpha)/k$ is abelian. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two lifts of $\gamma_1, \gamma_2 \in \Gamma$ to $\mathrm{Gal}(K(\alpha)/k)$. Remark that since

$$\left(\alpha^{\tilde{\gamma}_i - n_{\gamma_i}}\right)^m = a_{\gamma_i}^m,$$

there exists $\zeta_i \in \mu_m$ such that

$$\alpha^{\tilde{\gamma}_i - n_{\gamma_i}} = \zeta_i a_{\gamma_i}.$$

Now, we have

$$\begin{aligned}\left(\alpha^{\tilde{\gamma}_1 - n_{\gamma_1}}\right)^{\tilde{\gamma}_2 - n_{\gamma_2}} &= (\zeta_1 a_{\gamma_1})^{\tilde{\gamma}_2 - n_{\gamma_2}} \\ &= a_{\gamma_1}^{\tilde{\gamma}_2 - n_{\gamma_2}} \\ &= a_{\gamma_2}^{\tilde{\gamma}_1 - n_{\gamma_1}} \\ &= (\zeta_2 a_{\gamma_2})^{\tilde{\gamma}_1 - n_{\gamma_1}} \\ &= \left(\alpha^{\tilde{\gamma}_2 - n_{\gamma_2}}\right)^{\tilde{\gamma}_1 - n_{\gamma_1}},\end{aligned}$$

and thus

$$\left(\alpha^{\tilde{\gamma}_1}\right)^{\tilde{\gamma}_2} = \left(\alpha^{\tilde{\gamma}_2}\right)^{\tilde{\gamma}_1}.$$

We conclude that the extension is abelian.

Conversely, suppose that the extension $K(\alpha)/k$ is abelian. Going back to the proof of Theorem A.3, we gave an explicit formula for a_γ, namely

$$a_\gamma = \alpha^{\tilde{\gamma} - n_\gamma},$$

where $\tilde{\gamma}$ is any extension of γ to $K(\alpha)$. We then have

$$a_{\gamma_1}^{\gamma_2 - n_{\gamma_2}} = \left(\alpha^{\tilde{\gamma}_1 - n_{\gamma_1}}\right)^{\tilde{\gamma}_2 - n_{\gamma_2}} = \left(\alpha^{\tilde{\gamma}_2 - n_{\gamma_2}}\right)^{\tilde{\gamma}_1 - n_{\gamma_1}} = a_{\gamma_2}^{\gamma_1 - n_{\gamma_1}},$$

since the extension $K(\alpha)/k$ is abelian.

Q.E.D.

Appendix B

Numerical examples when the base field is $\mathbb{Q}$

Because of Theorem 5.27 and 5.23, we have to find examples for which there exist finite distinguished minimal cocyclic subgroups. For each of the examples below, we give a set S which is a 1-cover and the corresponding S_{min} which have been found with the help of a computer. In order to do so, we computed all minimal cocyclic subgroups of the Galois group and checked which ones are distinguished. We completed the set S with appropriate Frobenius automorphisms corresponding to some finite unramified primes.

If K is an abelian extension of $\mathbb{Q}$, we denote its Galois group by G. Given a prime p or ∞, we denote K^{G_p} more simply by K_p (not to be confused with a completion of K), and we denote $S \setminus \{p\}$ by R_p. Hence, $R_p \cup \{p\} = S$, and we also set $n_p = |G_p|$. Moreover, we let $\Gamma_p = G/G_p = \mathrm{Gal}(K_p/\mathbb{Q})$. For each of the field K_p where $p \in S_{min}$, we give the number of roots of unity in K_p, the cardinality of G_p, the degree of K_p over $\mathbb{Q}$, the R_p-equivariant L-function of $K_p/\mathbb{Q}$ evaluated at 0 which is an element of $\mathbb{Q}[\Gamma_p]$ by a theorem of Siegel, and the greatest common divisor of the coefficients of $\theta_{K_p/\mathbb{Q},R_p}(0)$ if it already lies in $\mathbb{Z}[\Gamma_p]$. Given any $\alpha \in \mathbb{Z}[G]$, we will denote the gcd of the coefficients of α by $\gamma(\alpha)$. For each of the field K_p when p is a finite prime, we choose a sample of primes t such that $\{t\} \cap S = \varnothing$ and $\mu_{K,\{t\}} = 1$, and we check part (1) of Proposition 5.13 for the element $\theta_{K_p/\mathbb{Q},R_p,\{t\}}(0) \in \mathbb{Z}[\Gamma_p]$. If $\infty \in S_{min}$, we check that the positive Stark unit ε for $St(K_\infty/\mathbb{Q}, S, \infty)$ is a square in K_∞ and if $\varepsilon_\infty = \eta_\infty^2$ then we also check that η_∞ is 2-abelian over $\mathbb{Q}$, i.e. that $K_\infty(\sqrt{\eta_\infty})/\mathbb{Q}$ is an abelian extension of number fields.

B.1 Full cyclotomic fields $\mathbb{Q}(\zeta_m)$.

The following table shows all $m \not\equiv 2 \pmod 4$ such that $m \leq 200$ and for which there exists a finite distinguished minimal cocyclic subgroup. We provide an example

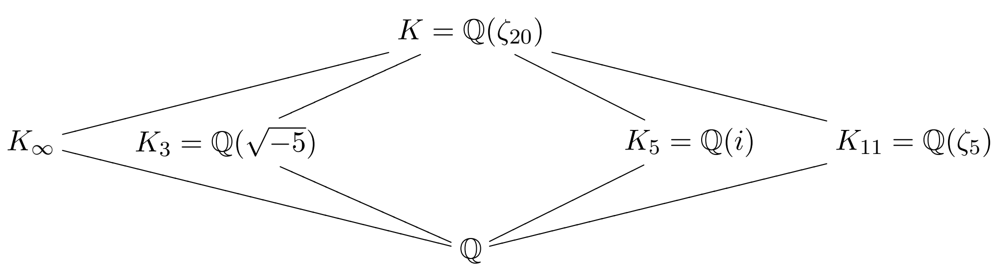
Table B.1: Full cyclotomic fields with finite distinguished subgroups.

m	Factorization	Structure of G_m	Distinguished place v
20	$2^2 \cdot 5$	$(4,2)$	5
24	$2^3 \cdot 3$	$(2,2,2)$	3
40	$2^3 \cdot 5$	$(4,2,2)$	5
48	$2^4 \cdot 3$	$(4,2,2)$	3
60	$2^2 \cdot 3 \cdot 5$	$(4,2,2)$	$2,3,5$
68	$2^2 \cdot 17$	$(16,2)$	17
80	$2^4 \cdot 5$	$(4,4,2)$	5
96	$2^5 \cdot 3$	$(8,2,2)$	3
120	$2^3 \cdot 3 \cdot 5$	$(4,2,2,2)$	$2,3,5$
136	$2^3 \cdot 17$	$(16,2,2)$	17
160	$2^5 \cdot 5$	$(8,4,2)$	5
171	$3^2 \cdot 19$	$(9,3,2,2)$	19
192	$2^6 \cdot 3$	$(16,2,2)$	3
195	$3 \cdot 5 \cdot 13$	$(3,4,4,2)$	5

for 10 of them.

B.1.1 The field $\mathbb{Q}(\zeta_{20})$.

Let $S = \{\infty, 2, 5, 3, 11\}$. In this case, $S_{min} = \{\infty, 3, 5, 11\}$ and we have the following diagram of fields



We computed the following data:

Table B.2: Data for the field $\mathbb{Q}(\zeta_{20})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_5	1	4	4	2	$[1, -1]$	1
K_3	2	2	4	2	$[2, -2]$	2
K_{11}	1	10	2	4	$[3/5, -1/5, 1/5, -3/5]$	-

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 4$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

Table B.3: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
7	$8 = 4 \cdot 2$
13	$12 = 4 \cdot 3$
17	$16 = 4 \cdot 4$
19	$20 = 4 \cdot 5$
23	$24 = 4 \cdot 6$
29	$28 = 4 \cdot 7$

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

The prime 5 is the only ramified prime, but since $h_{K_5} = 1$, we can conclude that $St(K/\mathbb{Q}, S, 5)$ is true by Theorem 5.24.

- Moreover, we made the following computation relative to K_∞. First of all, in $\mathbb{Q}(\zeta_{20})$, we have

$$(1 - \sigma_3^{-1})(1 - \sigma_{11}^{-1}) = 1 - \sigma_7 - \sigma_{11} + \sigma_{17}.$$

Hence, a Stark unit for $St(\mathbb{Q}(\zeta_{20})^+/\mathbb{Q}, \{\infty, 2, 5, 3, 11\}, \infty)$ is given by

$$\varepsilon_\infty = \frac{c_{20} \cdot c_{20}^{\sigma_{17}}}{c_{20}^{\sigma_7} \cdot c_{20}^{\sigma_{11}}}.$$

Its minimal polynomial over $\mathbb{Q}$ is

$$q(x) = x^4 - 164x^3 + 1606x^2 - 164x + 1.$$

We get $\varepsilon_\infty = \eta_\infty^2$ as we should by Theorem 5.26, where

$$\eta_\infty = \frac{1}{704}\varepsilon_\infty^3 - \frac{15}{64}\varepsilon_\infty^2 + \frac{165}{64}\varepsilon_\infty + \frac{45}{704},$$

and η_∞ is itself a square in K_∞, hence the abelian condition is also satisfied:

$$\eta_\infty = \left(\frac{17}{2816}\varepsilon_\infty^3 - \frac{253}{256}\varepsilon_\infty^2 + \frac{2417}{256}\varepsilon_\infty - \frac{973}{2816}\right)^2.$$

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 4$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3],$$

  but we already know this by Theorem 5.29. Moreover, we checked numerically that $BrSt(K/\mathbb{Q}, R_3)$ is true.

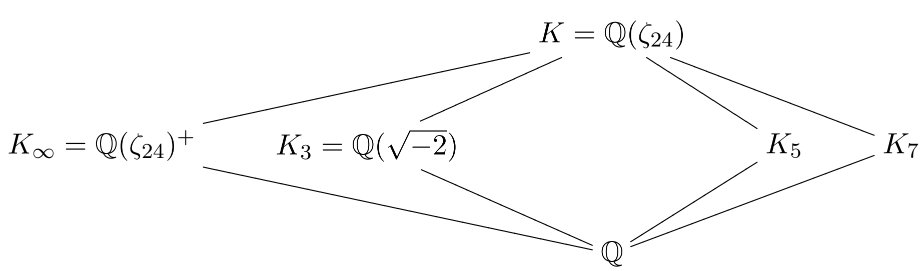
- In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus

$$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

  but we already know this by Theorem 5.29. Since $h_{K_{11}} = 1$, we conclude that $BrSt(K/\mathbb{Q}, R_{11})$ is true by Theorem 5.28.

B.1.2 The field $\mathbb{Q}(\zeta_{24})$.

Let $S = \{\infty, 2, 3, 5, 7\}$. In this case, $S_{min} = \{\infty, 3, 5, 7\}$ and we have the following diagram of fields



We computed the following data:

Table B.4: Data for the field $\mathbb{Q}(\zeta_{24})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_3	1	2	4	2	$[2, -2]$	2
K_5	2	4	2	4	$[1/2, -1/2, 1/2, -1/2]$	-
K_7	1	6	2	4	$[1/3, -1/3, 1/3, -1/3]$	-

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 4$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

From the table below, we see that we always have

Table B.5: The Stickelberger element for the ramified prime 3.

t	$\gamma(\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0))$
11	$20 = 4 \cdot 5$
13	$28 = 4 \cdot 7$
17	$32 = 4 \cdot 8$
19	$36 = 4 \cdot 9$
23	$48 = 4 \cdot 12$
29	$60 = 4 \cdot 15$
31	$64 = 4 \cdot 16$

$$\frac{\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

Moreover, since $h_{K_3} = 1$, we conclude that $St(K/\mathbb{Q}, S, 3)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$q(x) = x^2 - 14x + 1.$$

We also have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{1}{4}\varepsilon_\infty + \frac{1}{4}.$$

The algebraic number η_∞ is not a square in $\mathbb{Q}(\varepsilon_\infty)$, but we checked numerically that the abelian condition is satisfied.

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 2$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_5)$ is true.

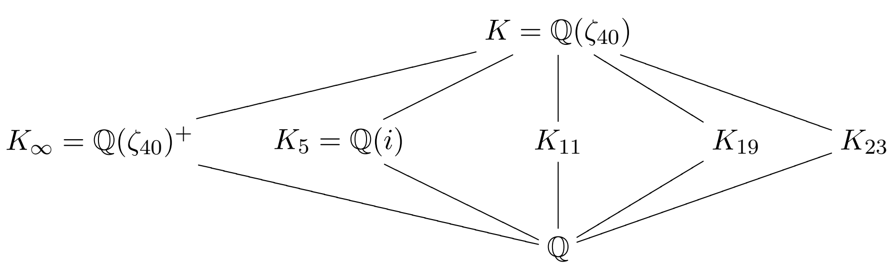
- In K_7 one has $\gamma(w_{K_7} \cdot \theta_{K_7/\mathbb{Q},R_7}(0)) = 2$, thus

$$\frac{w_{K_7} \cdot \theta_{K_7/\mathbb{Q},R_7}(0)}{n_7} \in \mathbb{Z}[\Gamma_7],$$

  but we already know this by Theorem 5.29. Moreover, since $h_{K_7} = 1$, we conclude that $BrSt(K/\mathbb{Q}, R_7)$ holds true because of Theorem 5.28.

B.1.3 The field $\mathbb{Q}(\zeta_{40})$.

Let $S = \{\infty, 2, 5, 11, 19, 23\}$. In this case, $S_{min} = \{\infty, 5, 11, 19, 23\}$ and we have the following diagram of fields



We computed the following data:

Table B.6: Data for the field $\mathbb{Q}(\zeta_{40})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_5	1	4	8	2	$[2, -2]$	2
K_{11}	1	10	2	8	$[3, 1, -1, -3, 1, -1, 3, -3]/5$	-
K_{19}	2	2	2	8	$[-1, -1, -1, 1, 1, 1, 1, -1]$	1
K_{23}	2	2	4	4	$[2, 2, -2, -2]$	2

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

Table B.7: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
3	8
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$
17	$32 = 8 \cdot 4$
3389	$6776 = 8 \cdot 847$

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

Moreover, since $h_{K_5} = 1$, we conclude that $St(K/\mathbb{Q}, S, 5)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$q(x) = x^2 - 1442x + 1.$$

We also have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{1}{38}\varepsilon_\infty + \frac{1}{38}.$$

Furthermore, η_∞ itself is a square in $\mathbb{Q}(\varepsilon_\infty)$:

$$\eta_\infty = \left(\frac{1}{228}\varepsilon_\infty - \frac{37}{228}\right)^2.$$

Hence, the abelian condition is satisfied.

- In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus

$$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

but we already know this by Theorem 5.29. Since $h_{K_{11}} = 1$, we also conclude that

$BrSt(K/\mathbb{Q}, R_{11})$ is true by Theorem 5.28.

- In K_{19} one has $\gamma(w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)) = 2$, thus

$$\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{19})$ is true.

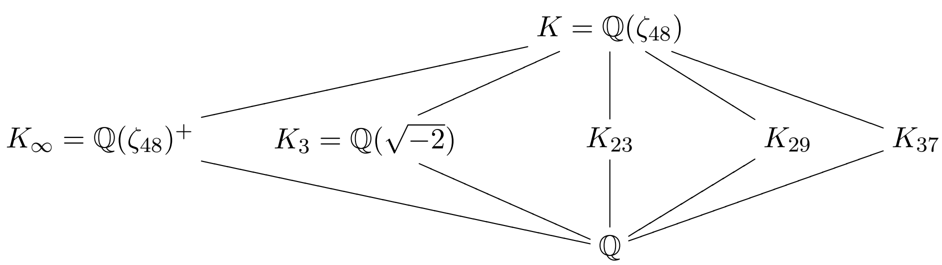
- In K_{23} one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 4$, thus

$$\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

B.1.4 The field $\mathbb{Q}(\zeta_{48})$.

Let $S = \{\infty, 2, 3, 23, 29, 37\}$. In this case, $S_{min} = \{\infty, 3, 23, 29, 37\}$ and we have the following diagram of fields



We computed the following data:

Table B.8: Data for the field $\mathbb{Q}(\zeta_{48})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_3	1	2	8	2	$[4, -4]$	4
K_{23}	2	2	2	8	$[0, 0, 2, 0, 0, 0, -2, 0]$	2
K_{29}	2	4	4	4	$[2, 2, -2, -2]$	2
K_{37}	1	12	4	4	$[2, -2, 2, -2]/3$	–

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 8$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

From the table below, we see that we always have

Table B.9: The Stickelberger element for the ramified prime 3.

t	$\gamma(\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0))$
7	$32 = 8 \cdot 4$
13	$56 = 8 \cdot 7$
17	$64 = 8 \cdot 8$
31	$128 = 8 \cdot 16$
3389	$13560 = 8 \cdot 1695$

$$\frac{\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

Moreover, since $h_{K_3} = 1$, we conclude that $St(K/\mathbb{Q}, S, 3)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$q(x) = x^8 - 11272\,x^7 + 5126172\,x^6 - 389502008\,x^5 + 2987561030\,x^4 - 389502008\,x^3 + 5126172\,x^2 - 11272\,x + 1.$$

We also have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{2474231}{1030863927296}\varepsilon_\infty^7 - \frac{3984220545}{147266275328}\varepsilon_\infty^6 + \frac{12683468746735}{1030863927296}\varepsilon_\infty^5 - \frac{963779376743095}{1030863927296}\varepsilon_\infty^4 + \frac{7396586738569165}{1030863927296}\varepsilon_\infty^3 - \frac{999765455993805}{1030863927296}\varepsilon_\infty^2 + \frac{2795083918475}{147266275328}\varepsilon_\infty + \frac{8116217995}{1030863927296}.$$

Furthermore, numerically we have that η_∞ is itself is a square in $\mathbb{Q}(\varepsilon_\infty)$. Hence, the abelian condition is satisfied.

- In K_{23} one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 4$, thus

$$\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

- In K_{29} one has $\gamma(w_{K_{29}} \cdot \theta_{K_{29}/\mathbb{Q},R_{29}}(0)) = 8$, thus

$$\frac{w_{K_{29}} \cdot \theta_{K_{29}/\mathbb{Q},R_{29}}(0)}{n_{29}} \in \mathbb{Z}[\Gamma_{29}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{29})$ is true.

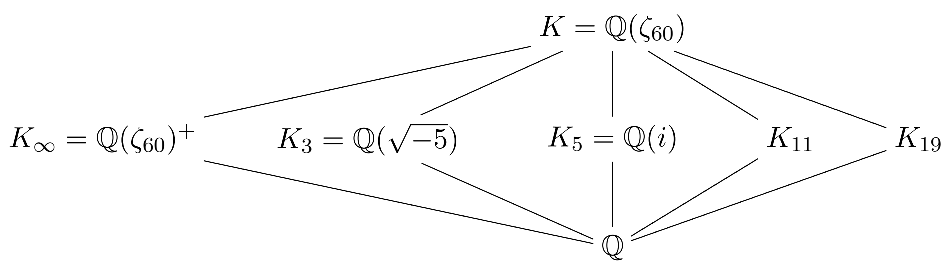
- In K_{37} one has $\gamma(w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)) = 8$, thus

$$\frac{w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)}{n_{37}} \in \mathbb{Z}[\Gamma_{37}],$$

but we already know this by Theorem 5.29. Since $h_{K_{37}} = 1$, we also conclude that $BrSt(K_{37}/\mathbb{Q}, R_{37})$ is true by Theorem 5.28.

B.1.5 The field $\mathbb{Q}(\zeta_{60})$.

Let $S = \{\infty, 2, 3, 5, 11, 19\}$. In this case, $S_{min} = \{\infty, 3, 5, 11, 19\}$ and we have the following diagram of fields



We computed the following data:

Table B.10: Data for the field $\mathbb{Q}(\zeta_{60})$ (First example).

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_3	2	2	8	2	$[4, -4]$	4
K_5	1	4	8	2	$[2, -2]$	2
K_{11}	2	10	2	8	$[3, -1, 1, -3, 3, -1, 1, -3]/5$	–
K_{19}	2	6	2	8	$[1, 1, -1, -1, 1, 1, -1, -1]/3$	–

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 8$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

From the table below, we see that we always have

Table B.11: The Stickelberger element for the ramified prime 3.

t	$\gamma(\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0))$
7	$24 = 8 \cdot 3$
13	$56 = 8 \cdot 7$
17	$72 = 8 \cdot 9$
31	$128 = 8 \cdot 16$
3389	$13552 = 8 \cdot 1694$

$$\frac{\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

Moreover, we checked numerically that $St(\mathbb{Q}(\zeta_{60})/\mathbb{Q}, S, 3)$ is true.

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

Table B.12: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$
17	$32 = 8 \cdot 4$
31	$64 = 8 \cdot 8$
3389	$6776 = 8 \cdot 847$

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

Moreover, since $h_{K_5} = 1$, we conclude that $St(\mathbb{Q}(\zeta_{60})/\mathbb{Q}, S, 5)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$q(x) = x^4 - 79\,x^3 + 201\,x^2 - 79\,x + 1.$$

We also have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{1}{209}\varepsilon_\infty^3 - \frac{80}{209}\varepsilon_\infty^2 + \frac{300}{209}\varepsilon_\infty + \frac{20}{209}.$$

Furthermore, η_∞ itself is a square in $\mathbb{Q}(\varepsilon_\infty)$:

$$\eta_\infty = \left(\frac{3}{209}\varepsilon_\infty^3 - \frac{701}{627}\varepsilon_\infty^2 + \frac{1009}{627}\varepsilon_\infty + \frac{199}{627}\right)^2.$$

Hence, the abelian condition is satisfied.

- In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus

$$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

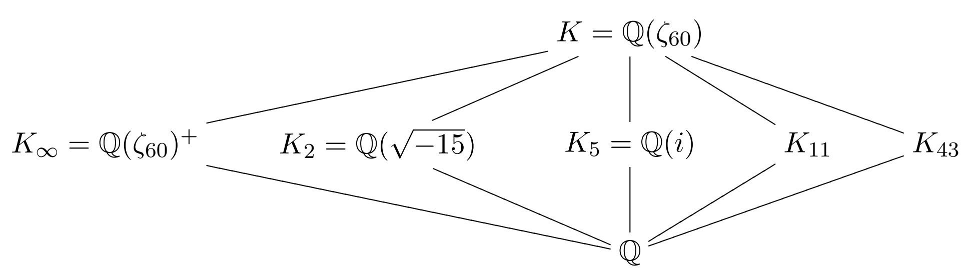
but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{11})$ is true.

- In K_{19} one has $\gamma(w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)) = 2$, thus

$$\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{19})$ is true.

Here is another example with the same field $\mathbb{Q}(\zeta_{60})$. Let $S = \{\infty, 2, 3, 5, 11, 43\}$. In this case, $S_{min} = \{\infty, 2, 5, 11, 43\}$ and we have the following diagram of fields



We computed the following data:

Table B.13: Data for the field $\mathbb{Q}(\zeta_{60})$ (Second example).

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_2	2	2	8	2	$[4, -4]$	4
K_5	1	4	8	2	$[2, -2]$	2
K_{11}	2	10	2	8	$[1, -7, -3, -1, 1, 3, 7, -1]/5$	–
K_{43}	2	6	4	4	$[2, -2, 2, -2]/3$	–

- In K_2 one has $\gamma(w_{K_2} \cdot \theta_{K_2/\mathbb{Q},R_2}(0)) = 8$, thus

$$\frac{w_{K_2} \cdot \theta_{K_2/\mathbb{Q},R_2}(0)}{n_2} \in \mathbb{Z}[\Gamma_2].$$

From the table below, we see that we always have

Table B.14: The Stickelberger element for the ramified prime 2.

t	$\gamma(\theta_{K_2/\mathbb{Q},R_2,\{t\}}(0))$
7	$32 = 8 \cdot 4$
13	$56 = 8 \cdot 7$
17	$64 = 8 \cdot 8$
31	$120 = 8 \cdot 15$
3389	$13560 = 8 \cdot 1695$

$$\frac{\theta_{K_2/\mathbb{Q},R_2,\{t\}}(0)}{n_2} \in \mathbb{Z}[\Gamma_2].$$

Moreover, we checked numerically that $St(\mathbb{Q}(\zeta_{60})/\mathbb{Q}, S, 2)$ holds true.

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

Moreover, since $h_{K_5} = 1$, we conclude that $St(K/\mathbb{Q}, S, 5)$ is true by Theorem 5.24.

Table B.15: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$
17	$32 = 8 \cdot 4$
31	$64 = 8 \cdot 8$
3389	$6776 = 8 \cdot 847$

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

  $$q(x) = x^8 - 538\,x^7 + 14763\,x^6 - 71876\,x^5 + 118325\,x^4 - 71876\,x^3 + 14763\,x^2 - 538\,x + 1,$$

  hence $\mathbb{Q}(\varepsilon_\infty) = K_\infty$. We also have $\varepsilon_\infty = \eta_\infty^2$ where

  $$\eta_\infty = \frac{24919}{130332851}\varepsilon_\infty^7 - \frac{53631771}{521331404}\varepsilon_\infty^6 + \frac{12188357}{4308524}\varepsilon_\infty^5 - \frac{3627530725}{260665702}\varepsilon_\infty^4 + \frac{6129121873}{260665702}\varepsilon_\infty^3 - \frac{33221103}{2154262}\varepsilon_\infty^2 + \frac{2404381171}{521331404}\varepsilon_\infty + \frac{18410903}{521331404}.$$

  In this case, η_∞ itself is not a square in K_∞, but we checked that the abelian condition is satisfied, i.e. $K_\infty(\eta_\infty^{1/2})/\mathbb{Q}$ is an abelian extension of number fields.

- In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus

  $$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{11})$ is true.

- In K_{43} one has $\gamma(w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)) = 4$, thus

  $$\frac{w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}],$$

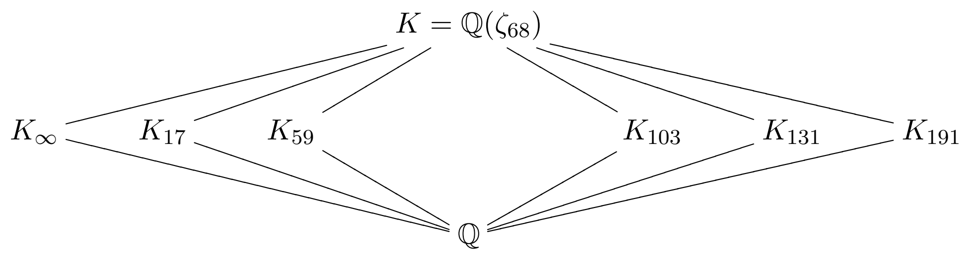
  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{43})$ is true.

B.1.6 The field $\mathbb{Q}(\zeta_{68})$.

Let $S = \{\infty, 2, 17, 59, 103, 131, 191\}$. In this case

$$S_{min} = \{\infty, 17, 59, 103, 131, 191\},$$

and we have the following diagram of fields



We computed the following data:

Table B.16: Data for the field $\mathbb{Q}(\zeta_{68})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	gcd
$K_{17} = \mathbb{Q}(i)$	1	4	16	2	4
K_{59}	4	2	8	4	8
K_{103}	1	34	2	16	–
$K_{131} = \mathbb{Q}(\sqrt{-17})$	4	2	16	2	16
K_{191}	4	2	4	8	4

Table B.17: Data for the field $\mathbb{Q}(\zeta_{68})$.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
17	$[4, -4]$
59	$[0, 8, -8, 0]$
103	$[8, -20, 22, 6, -18, 10, -2, -4, 4, 2, -10, 18, -6, -22, 20, -8]/17$
131	$[16, -16]$
191	$[-4, -4, 0, -4, 0, 4, 4, 4]$

- In K_{17} one has $\gamma(w_{K_{17}} \cdot \theta_{K_{17}/\mathbb{Q},R_{17}}(0)) = 16$, thus

$$\frac{w_{K_{17}} \cdot \theta_{K_{17}/\mathbb{Q},R_{17}}(0)}{n_{17}} \in \mathbb{Z}[\Gamma_{17}].$$

From the table below, we see that we always have

Table B.18: The Stickelberger element for the ramified prime 17.

t	$\gamma(\theta_{K_{17}/\mathbb{Q},R_{17},\{t\}}(0))$
7	$32 = 16 \cdot 2$
13	$48 = 16 \cdot 3$
29	$112 = 16 \cdot 7$
31	$128 = 16 \cdot 8$
3389	$13552 = 16 \cdot 847$

$$\frac{\theta_{K_{17}/\mathbb{Q},R_{17},\{t\}}(0)}{n_{17}} \in \mathbb{Z}[\Gamma_{17}].$$

Moreover, since $h_{K_{17}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 17)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$\begin{aligned} q(x) = {} & x^{16} - 20764496\,x^{15} + 1389588297208\,x^{14} - 5046316108334576\,x^{13} + \\ & 4873057566093211164\,x^{12} - 1208111469114746731088\,x^{11} + \\ & 78671135798960222208456\,x^{10} - 370087576718887268362352\,x^{9} + \\ & 587242444047744287908934\,x^{8} - 370087576718887268362352\,x^{7} + \\ & 78671135798960222208456\,x^{6} - 1208111469114746731088\,x^{5} + \\ & 4873057566093211164\,x^{4} - 5046316108334576\,x^{3} + 1389588297208\,x^{2} - \\ & 20764496\,x + 1. \end{aligned}$$

We also have $\varepsilon_\infty = \eta_\infty^2$ for some η_∞ which we will not write down. Furthermore, we checked that η_∞ itself is a square in $\mathbb{Q}(\varepsilon_\infty)$. Hence, the abelian condition is satisfied.

- In K_{59} one has $\gamma(w_{K_{59}} \cdot \theta_{K_{59}/\mathbb{Q},R_{59}}(0)) = 16$, thus

$$\frac{w_{K_{59}} \cdot \theta_{K_{59}/\mathbb{Q},R_{59}}(0)}{n_{59}} \in \mathbb{Z}[\Gamma_{59}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{59})$ is true.

- In K_{103} one has $\gamma(w_{K_{103}} \cdot \theta_{K_{103}/\mathbb{Q},R_{103}}(0)) = 4$, thus

$$\frac{w_{K_{103}} \cdot \theta_{K_{103}/\mathbb{Q},R_{103}}(0)}{n_{103}} \in \mathbb{Z}[\Gamma_{103}],$$

  but we already know this by Theorem 5.29. Since $h_{K_{103}} = 1$, we also conclude that $BrSt(K/\mathbb{Q}, R_{103})$ is true by Theorem 5.28.

- In K_{131} one has $\gamma(w_{K_{131}} \cdot \theta_{K_{131}/\mathbb{Q},R_{131}}(0)) = 32$, thus

$$\frac{w_{K_{131}} \cdot \theta_{K_{131}/\mathbb{Q},R_{131}}(0)}{n_{131}} \in \mathbb{Z}[\Gamma_{131}],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{131})$ is true.

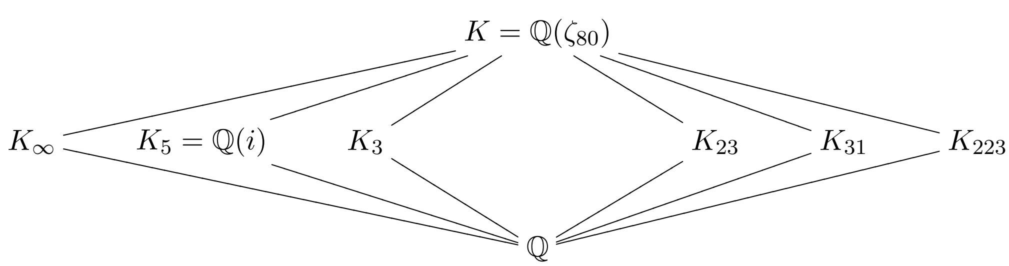
- In K_{191} one has $\gamma(w_{K_{191}} \cdot \theta_{K_{191}/\mathbb{Q},R_{191}}(0)) = 8$, thus

$$\frac{w_{K_{191}} \cdot \theta_{K_{191}/\mathbb{Q},R_{191}}(0)}{n_{191}} \in \mathbb{Z}[\Gamma_{191}],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{191})$ is true.

B.1.7 The field $\mathbb{Q}(\zeta_{80})$.

Let $S = \{\infty, 2, 3, 5, 23, 31, 223\}$. In this case, $S_{min} = \{\infty, 3, 5, 23, 31, 223\}$ and we have the following diagram of fields



We computed the following data:

Table B.19: Data for the field $\mathbb{Q}(\zeta_{80})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	gcd
K_5	1	4	16	2	4
K_3	20	2	4	8	2
K_{23}	2	2	4	8	2
K_{31}	5	10	2	16	–
K_{223}	2	2	4	8	2

Table B.20: Data for the field $\mathbb{Q}(\zeta_{80})$.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
5	$[4, -4]$
3	$[2, -2, 2, -2, 2, -2, 2, -2]$
23	$[2, 2, -2, -2, 2, 2, -2, -2]$
31	$[-3, -1, 1, 3, -3, -9, -7, -1, 1, 9, -3, -1, 3, 7, 3, 1]/5$
223	$[-2, -2, 2, -2, 2, 2, 2, -2]$

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 16$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

Table B.21: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
7	$32 = 16 \cdot 2$
13	$48 = 16 \cdot 3$
17	$64 = 16 \cdot 4$
29	$112 = 16 \cdot 7$
3389	$13552 = 16 \cdot 847$

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

Moreover, since $h_{K_5} = 1$, we conclude that $St(K/\mathbb{Q}, S, 5)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$
\begin{aligned}
q(x) = {} & x^{16} - 104256\,x^{15} + 250074680\,x^{14} - 194823768000\,x^{13} + \\
& 59914703898780\,x^{12} - 6707694294719808\,x^{11} + 280735144193463688\,x^{10} - \\
& 4512975958112256960\,x^{9} + 22116392924226983750\,x^{8} - \\
& 4512975958112256960\,x^{7} + 280735144193463688\,x^{6} - \\
& 6707694294719808\,x^{5} + 59914703898780\,x^{4} - 194823768000\,x^{3} + \\
& 250074680\,x^{2} - 104256\,x + 1.
\end{aligned}
$$

  We also have $\varepsilon_\infty = \eta_\infty^2$ for some η_∞ which we checked is itself a square in $\mathbb{Q}(\varepsilon_\infty)$. Hence, the abelian condition is satisfied.

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 4$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_3)$ is true.

- In K_{23} one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 4$, thus

$$\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

- In K_{31} one has $\gamma(w_{K_{31}} \cdot \theta_{K_{31}/\mathbb{Q},R_{31}}(0)) = 2$, thus

$$\frac{w_{K_{31}} \cdot \theta_{K_{31}/\mathbb{Q},R_{31}}(0)}{n_{31}} \in \mathbb{Z}[\Gamma_{31}],$$

  but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{31})$ is true.

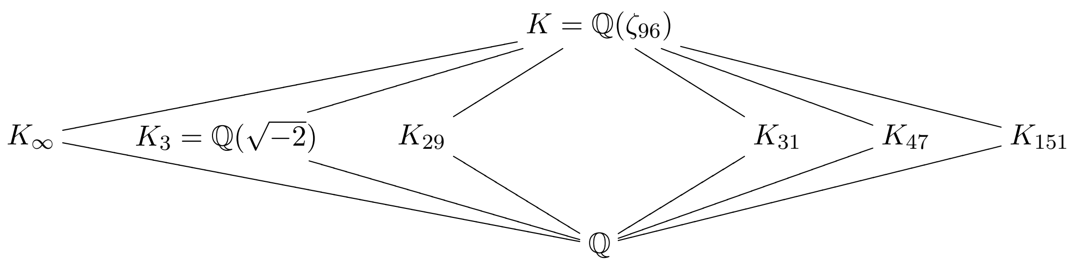
- In K_{223} one has $\gamma(w_{K_{223}} \cdot \theta_{K_{223}/\mathbb{Q},R_{223}}(0)) = 4$, thus

$$\frac{w_{K_{223}} \cdot \theta_{K_{223}/\mathbb{Q},R_{223}}(0)}{n_{223}} \in \mathbb{Z}[\Gamma_{223}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{223})$ is true.

B.1.8 The field $\mathbb{Q}(\zeta_{96})$.

Let $S = \{\infty, 2, 3, 29, 31, 47, 151\}$. In this case, $S_{min} = \{\infty, 3, 29, 31, 47, 151\}$ and we have the following diagram of fields



We computed the following data:

Table B.22: Data for the field $\mathbb{Q}(\zeta_{96})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	gcd
K_3	1	2	16	2	8
K_{29}	2	4	8	4	2
K_{31}	9	6	2	16	1
K_{47}	18	2	2	16	1
K_{151}	1	6	4	8	2

Table B.23: Data for the field $\mathbb{Q}(\zeta_{96})$.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
3	$[8, -8]$
29	$[2, -2, 2, -2]$
31	$[-1, -1, 1, -1, -1, 1, -1, 1, 1, 1, -1, 1, -1, 1, -1, 1]$
47	$[1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1]$
151	$[2, -2, 2, 2, 2, -2, -2, -2]$

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 16$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

From the table below, we see that we always have

Table B.24: The Stickelberger element for the ramified prime 3.

t	$\gamma(\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0))$
7	$64 = 16 \cdot 4$
13	$112 = 16 \cdot 7$
17	$128 = 16 \cdot 8$
19	$144 = 16 \cdot 9$
3389	$27120 = 16 \cdot 1695$

$$\frac{\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

Moreover, since $h_{K_3} = 1$, we conclude that $St(K/\mathbb{Q}, S, 3)$ is true by Theorem 5.24.

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$\begin{aligned} q(x) = {} & x^8 - 5325320\,x^7 + 454281145372\,x^6 - 4689184218766904\,x^5 + \\ & 363558074501621830\,x^4 - 4689184218766904\,x^3 + 454281145372\,x^2 - \\ & 5325320\,x + 1, \end{aligned}$$

and thus $\mathbb{Q}(\varepsilon_\infty) = K_\infty$. We also have $\varepsilon_\infty = \eta_\infty^2$ for some $\eta_\infty \in \mathbb{Q}(\varepsilon_\infty)$ which is itself a square in $\mathbb{Q}(\varepsilon_\infty)$. Hence, the abelian condition is satisfied.

- In K_{29} one has $\gamma(w_{K_{29}} \cdot \theta_{K_{29}/\mathbb{Q},R_{29}}(0)) = 8$, thus

$$\frac{w_{K_{29}} \cdot \theta_{K_{29}/\mathbb{Q},R_{29}}(0)}{n_{29}} \in \mathbb{Z}[\Gamma_{29}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{29})$ is true.

- In K_{31} one has $\gamma(w_{K_{31}} \cdot \theta_{K_{31}/\mathbb{Q},R_{31}}(0)) = 6$, thus

$$\frac{w_{K_{31}} \cdot \theta_{K_{31}/\mathbb{Q},R_{31}}(0)}{n_{31}} \in \mathbb{Z}[\Gamma_{31}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{31})$ is true.

- In K_{47} one has $\gamma(w_{K_{47}} \cdot \theta_{K_{47}/\mathbb{Q},R_{47}}(0)) = 2$, thus

$$\frac{w_{K_{47}} \cdot \theta_{K_{47}/\mathbb{Q},R_{47}}(0)}{n_{47}} \in \mathbb{Z}[\Gamma_{47}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{47})$ is true.

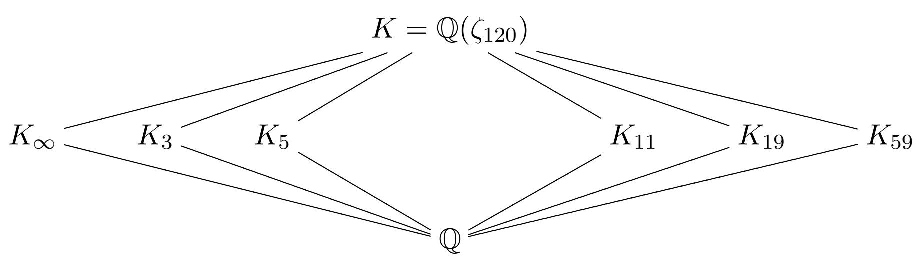
- In K_{151} one has $\gamma(w_{K_{151}} \cdot \theta_{K_{151}/\mathbb{Q},R_{151}}(0)) = 12$, thus

$$\frac{w_{K_{151}} \cdot \theta_{K_{151}/\mathbb{Q},R_{151}}(0)}{n_{151}} \in \mathbb{Z}[\Gamma_{151}],$$

but we already know this by Theorem 5.29. Since $h_{K_{151}} = 1$, we also conclude that $BrSt(K_{151}/\mathbb{Q}, R_{151})$ is true by Theorem 5.28.

B.1.9 The field $\mathbb{Q}(\zeta_{120})$.

Let $S = \{\infty, 2, 3, 5, 11, 19, 59\}$. In this case, $S_{min} = \{\infty, 3, 5, 11, 19, 59\}$ and we have the following diagram of fields



We computed the following data:

Table B.25: Data for the field $\mathbb{Q}(\zeta_{120})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_3	2	2	8	4	4
K_5	2	4	8	4	2
K_{11}	8	10	2	16	–
K_{19}	4	6	2	16	–
K_{59}	8	2	2	16	1

Table B.26: Data for the field $\mathbb{Q}(\zeta_{120})$.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
3	$[4, 4, -4, -4]$
5	$[2, -2, 2, -2]$
11	$[3, -1, 1, -1, -3, -3, 3, -1, 3, 1, -3, 1, 3, 1, -1, -3]/5$
19	$[1, 1, -1, -1, -1, -1, 1, 1, -1, 1, -1, 1, 1, 1, -1, -1]/3$
59	$[-1, -1, 1, 1, -1, 1, -1, 1, 1, 1, 1, -1, -1, 1, -1, 1, -1]$

- In K_3 one has $\gamma(w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)) = 8$, thus

$$\frac{w_{K_3} \cdot \theta_{K_3/\mathbb{Q},R_3}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

From the table below, we see that we always have

Table B.27: The Stickelberger element for the ramified prime 3.

t	$\gamma(\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0))$
241	$960 = 8 \cdot 120$
601	$2400 = 8 \cdot 300$
7	$24 = 8 \cdot 3$
13	$56 = 8 \cdot 7$

$$\frac{\theta_{K_3/\mathbb{Q},R_3,\{t\}}(0)}{n_3} \in \mathbb{Z}[\Gamma_3].$$

Moreover, we checked numerically that $St(\mathbb{Q}(\zeta_{120})/\mathbb{Q}, S, 3)$ is true.

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

Moreover, we checked numerically that $St(\mathbb{Q}(\zeta_{120})/\mathbb{Q}, S, 5)$ is true.

Table B.28: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
241	$480 = 8 \cdot 60$
601	$1200 = 8 \cdot 150$
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$

- In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$q(x) = x^2 - 34\,x + 1.$$

We also have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{1}{6}\varepsilon_\infty + \frac{1}{6}.$$

Furthermore, η_∞ itself is a square in $\mathbb{Q}(\varepsilon_\infty)$:

$$\eta_\infty = \left(\frac{1}{12}\varepsilon_\infty - \frac{5}{12}\right)^2.$$

Hence, the abelian condition is satisfied.

- In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus

$$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{11})$ is true except the abelian condition since the computation was too lengthy.

- In K_{19} one has $\gamma(w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)) = 2$, thus

$$\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{19})$ is true.

- In K_{59} one has $\gamma(w_{K_{59}} \cdot \theta_{K_{59}/\mathbb{Q},R_{59}}(0)) = 2$, thus

$$\frac{w_{K_{59}} \cdot \theta_{K_{59}/\mathbb{Q},R_{59}}(0)}{n_{59}} \in \mathbb{Z}[\Gamma_{59}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{59})$ is true.

B.1.10 The field $\mathbb{Q}(\zeta_{171})$.

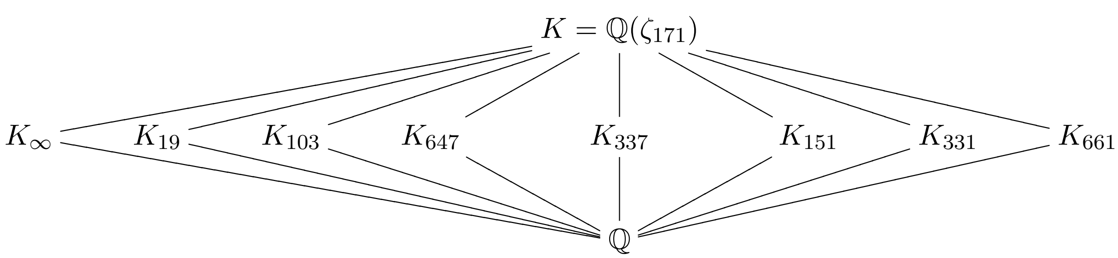
This is the first example of a full cyclotomic whose degree is not a power of 2 and which has a finite distinguished minimal cocyclic subgroup. Its Galois group is isomorphic to

$$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Let

$$S = \{\infty, 3, 19, 103, 647, 151, 331, 661, 337\}.$$

In this case, $S_{min} = \{\infty, 19, 103, 647, 151, 331, 661, 337\}$ and we have the following diagram of fields



We computed the following data:

Table B.29: Data for the field $\mathbb{Q}(\zeta_{171})$.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\gamma(\theta_{K_p/\mathbb{Q},R_p}(0))$
K_{19}	1	18	18	6	6
K_{103}	2943	6	6	18	6
K_{647}	333172	38	2	54	–
K_{151}	9	6	6	18	6
K_{331}	513	6	6	18	6
K_{661}	9	6	18	6	18
K_{337}	9	6	18	6	18

See the table below for the elements $\theta_{K_p/\mathbb{Q},R_p}(0)$.

- In K_{19} one has $\gamma(w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)) = 108 = 18 \cdot 6$, thus

$$\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}].$$

From the table below, we see that we always have

Table B.30: The Stickelberger element for the ramified prime 19.

t	$\gamma(\theta_{K_{19}/\mathbb{Q},R_{19},\{t\}}(0))$
5	18
7	18
11	18
13	18
29	18
31	18

$$\frac{\theta_{K_{19}/\mathbb{Q},R_{19},\{t\}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}].$$

Since $h_{K_{19}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 19)$ is true by Theorem 5.24.

- We also checked numerically that ε_∞ is a square in K_∞, but we did not check the abelian condition since the computation was too lengthy.

- In K_{103}, one has $\gamma(w_{K_{103}} \cdot \theta_{K_{103}/\mathbb{Q},R_{103}}(0)) = 36 = 6 \cdot 6$, thus we have

$$\frac{w_{K_{103}} \cdot \theta_{K_{103}/\mathbb{Q},R_{103}}(0)}{n_{103}} \in \mathbb{Z}[\Gamma_{103}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We checked that

$$\mathfrak{p}_i^{\frac{\theta_{K_{103}/\mathbb{Q},R_{103}}(0)}{6}}$$

is principal and generated by an anti-unit (for $i = 1, 2$). Thus $BrSt(K/\mathbb{Q}, R_{103})$ is true numerically. Note that in this case the abelian condition follows automatically

Table B.31: Data for the field $\mathbb{Q}(\zeta_{171})$.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
19	$[12, 6, -6, 6, -6, -12]$
103	$[0, 6, 18, 6, -6, -6, 0, -6, 6, -18, 6, -6, -12, 12, 6, 0, 0, -6]$
647	$[46, 4, 40, 32, 50, 20, 16, 18, 24, -32, 48, 72, 46, -48, 14, 88, -16, -22, -12, -2, 10, -46, 42, -6, 14, 22, 56, 6, -40,$ $-46, 8, 2, -6, -18, -60, -14, 6, -4, -48, 40, 60, -50, -14, -8, -10, -56, -88, -42, -72, 12, -20, 48, -24, -40]/19$
151	$[0, 6, -6, 0, -6, -6, 6, 0, 6, 0, 6, 6, -6, 0, -6, -6, 0, 6]$
331	$[0, -6, 0, 0, 6, -6, 6, 0, 0, 6, 6, 0, -6, -6, 6, -6, -6, 6]$
661	$[0, 0, -18, 18, 18, -18]$
337	$[-18, 0, 18, 18, 0, -18]$

since the element of the group ring we are interested in is

$$\theta_{K_{103}/\mathbb{Q},R_{103}}(0),$$

and $w_{K_{103}} = 6$.

- In K_{647}, one has $\gamma(w_{K_{647}} \cdot \theta_{K_{647}/\mathbb{Q},R_{647}}(0)) = 4 = 2 \cdot 2$, thus we have

$$\frac{w_{K_{647}} \cdot \theta_{K_{647}/\mathbb{Q},R_{647}}(0)}{n_{647}} \in \mathbb{Z}[\Gamma_{647}],$$

but we already know this by Theorem 5.29. We did not do the computations here, since they were too lengthy.

- In K_{151}, one has $\gamma(w_{K_{151}} \cdot \theta_{K_{151}/\mathbb{Q},R_{151}}(0)) = 36 = 6 \cdot 6$, thus we have

$$\frac{w_{K_{151}} \cdot \theta_{K_{151}/\mathbb{Q},R_{151}}(0)}{n_{151}} \in \mathbb{Z}[\Gamma_{151}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of one prime $\mathfrak{p}$. We checked that

$$\mathfrak{p}_{\mathfrak{i}}^{\frac{\theta_{K_{151}/\mathbb{Q},R_{151}}(0)}{6}}$$

is principal and generated by an anti-unit. This implies that $BrSt(K/\mathbb{Q}, R_{151})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$\theta_{K_{151}/\mathbb{Q},R_{151}}(0),$$

and $w_{K_{151}} = 6$.

- In K_{331}, one has $\gamma(w_{K_{331}} \cdot \theta_{K_{331}/\mathbb{Q},R_{331}}(0)) = 36 = 6 \cdot 6$, thus we have

$$\frac{w_{K_{331}} \cdot \theta_{K_{331}/\mathbb{Q},R_{331}}(0)}{n_{331}} \in \mathbb{Z}[\Gamma_{331}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. For $i = 1, 2$, we checked that

$$\mathfrak{p}_{\mathfrak{i}}^{\frac{\theta_{K_{331}/\mathbb{Q},R_{331}}(0)}{6}}$$

is principal and generated by an anti-unit. This implies that $BrSt(K/\mathbb{Q}, R_{331})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$\theta_{K_{331}/\mathbb{Q},R_{331}}(0),$$

and $w_{K_{331}} = 6$.

- In K_{661}, one has $\gamma(w_{K_{661}} \cdot \theta_{K_{661}/\mathbb{Q},R_{661}}(0)) = 108 = 18 \cdot 6$, thus we have

$$\frac{w_{K_{661}} \cdot \theta_{K_{661}/\mathbb{Q},R_{661}}(0)}{n_{661}} \in \mathbb{Z}[\Gamma_{661}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We checked that

$$\mathfrak{p}_\mathfrak{i}^{\frac{\theta_{K_{661}/\mathbb{Q},R_{661}}(0)}{18}}$$

is principal and generated by an anti-unit (for $i = 1, 2$). Thus $BrSt(K/\mathbb{Q}, R_{661})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$\frac{\theta_{K_{661}/\mathbb{Q},R_{661}}(0)}{3},$$

and $w_{K_{661}} = 6$.

- In K_{337}, one has $\gamma(w_{K_{337}} \cdot \theta_{K_{337}/\mathbb{Q},R_{337}}(0)) = 108 = 18 \cdot 6$, thus we have

$$\frac{w_{K_{337}} \cdot \theta_{K_{337}/\mathbb{Q},R_{337}}(0)}{n_{337}} \in \mathbb{Z}[\Gamma_{337}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We checked that

$$\mathfrak{p}_\mathfrak{i}^{\frac{\theta_{K_{337}/\mathbb{Q},R_{337}}(0)}{18}}$$

is principal and generated by an anti-unit (for $i = 1, 2$). Thus $BrSt(K/\mathbb{Q}, R_{337})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$\frac{\theta_{K_{337}/\mathbb{Q},R_{337}}(0)}{3},$$

and $w_{K_{337}} = 6$.

B.2 General abelian extensions of $\mathbb{Q}$.

B.2.1 Example 1

Let us come back to the example given in Section 4.5 and take the pair $(5, 31)$ and $l = 37$. Recall that $S = \{\infty, 5, 31, 37\}$ and $S_{min} = \{\infty, 5, 37\}$. We computed the following data

Table B.32: Data for example 1.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_5	3	2	2	10	$[1, -1, -1, 1, -1, 1, -1, 1, 1, -1]$	1
K_{37}	64	2	2	10	$[0, 0, 0, 2, 0, -2, 0, 0, 0, 0]$	2

- In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 2$, thus

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

From the table below, we see that we always have

Table B.33: The Stickelberger element for the ramified prime 5.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
3	2
7	2
11	2
13	2
17	2

$$\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5].$$

We also checked numerically that $St(K/\mathbb{Q}, S, 5)$ is true.

- In K_∞, we computed the minimal polynomial of ε_∞, and we obtained

$$p(X) = x^{10} - 242015x^9 + 449115185x^8 - 108113455735x^7 + 4075762267865x^6 - 40560353573727x^5 + 4075762267865x^4 - 108113455735x^3 + 449115185x^2 - 242015x + 1.$$

We also checked that $\varepsilon_\infty = \eta_\infty^2$ for some $\eta_\infty \in K_\infty$. Moreover, we checked that η_∞ is 2-abelian over $\mathbb{Q}$.

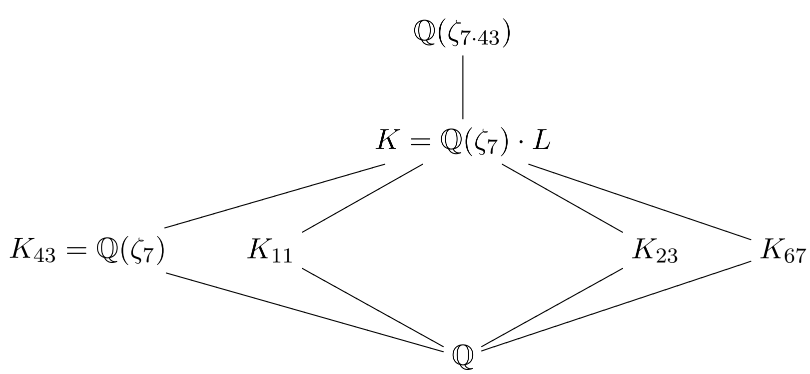
- In K_{37}, one has $\gamma(w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)) = 4$, thus we have

$$\frac{w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)}{n_{37}} \in \mathbb{Z}[\Gamma_{37}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{37})$ is true.

B.2.2 Example 2

Take the primes 7 and 43. Since $43 \equiv 1 \pmod 3$, let L be the unique subfield of $\mathbb{Q}(\zeta_{43})$ which is of degree 3 over $\mathbb{Q}$ and set $K = \mathbb{Q}(\zeta_7) \cdot L$. It is an abelian extension of $\mathbb{Q}$ with Galois group $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. There are 4 minimal cocyclic subgroups by Theorem 4.28 and they are precisely the 4 subgroups of order 3. The set $S = \{\infty, 7, 43, 11, 23, 67\}$ is a 1-cover and $S_{min} = \{43, 11, 23, 67\}$. There are only 2 ramified primes in K, namely 7 and 43. We thus have the following diagram of fields.



We computed the following data

Table B.34: Data for example 2.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_{43}	1	14	3	6	$[3, -9, -6, 6, 9, -3]/7$	–
K_{11}	57	2	3	6	$[-6, -15, -9, 9, 15, 6]$	3
K_{23}	9	2	3	6	$[-3, 0, 0, 3, 3, -3]$	3
K_{67}	9	2	3	6	$[3, -3, -3, 0, 0, 3]$	3

- In K_{43} one has $\gamma(w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)) = 6 = 2 \cdot 3$, thus

$$\frac{w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}].$$

From the table below, we see that we always have

Table B.35: The Stickelberger element for the ramified prime 43.

t	$\gamma(\theta_{K_{43}/\mathbb{Q},R_{43},\{t\}}(0))$
3	3
5	3
13	3
17	3
29	12
31	3
37	3

$$\frac{\theta_{K_{43}/\mathbb{Q},R_{43},\{t\}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}].$$

Moreover, since $h_{K_{43}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 43)$ is true by Theorem 5.24.

- In K_{11}, one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 6$, thus we have

$$\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{11})$ is true.

- In K_{23}, one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 6$, thus we have

$$\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}],$$

but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

- In K_{67}, one has $\gamma(w_{K_{67}} \cdot \theta_{K_{67}/\mathbb{Q},R_{67}}(0)) = 6$, thus we have

$$\frac{w_{K_{67}} \cdot \theta_{K_{67}/\mathbb{Q},R_{67}}(0)}{n_{67}} \in \mathbb{Z}[\Gamma_{67}],$$

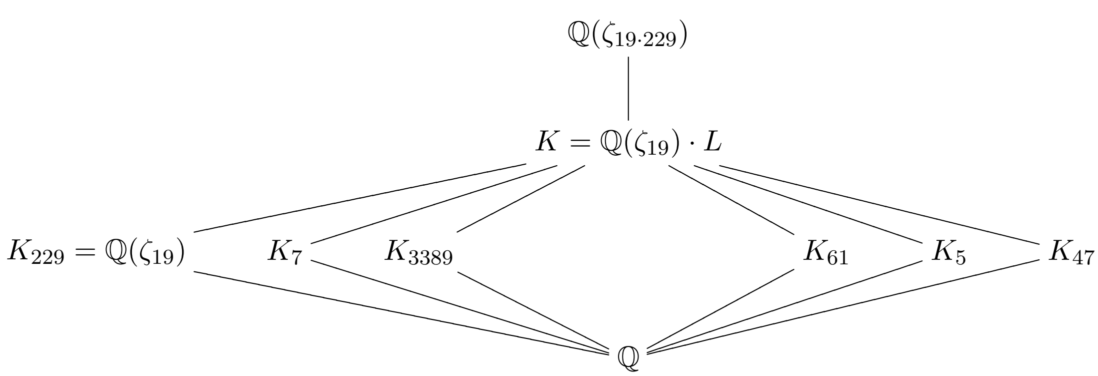
but we already know this by Theorem 5.29. We also checked numerically that $BrSt(K/\mathbb{Q}, R_{67})$ is true.

B.2.3 Example 3

Take the primes 19 and 229. Since $229 \equiv 1 \pmod 3$, let L be the unique subfield of $\mathbb{Q}(\zeta_{229})$ which is of degree 3 over $\mathbb{Q}$ and set $K = \mathbb{Q}(\zeta_{19}) \cdot L$. It is an abelian extension of $\mathbb{Q}$ with Galois group

$$G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

There are 6 minimal cocyclic subgroups: Three of order 3 and three of order 9. The set $S = \{\infty, 19, 229, 61, 7, 3389, 5, 47\}$ is a 1-cover and $S_{min} = \{229, 7, 3389, 61, 5, 47\}$. There are only 2 ramified primes in K, namely 19 and 229. We have the following diagram of fields.



We computed the following data

Table B.36: Data for example 3.

Fields	h_{K_p}	w_{K_p}	$[K:K_p]$	$[K_p:\mathbb{Q}]$	gcd
K_{229}	1	38	3	18	–
K_7	7151733	2	3	18	3
K_{3389}	3245751	2	3	18	3
K_{61}	63	2	9	6	9
K_5	3024	2	9	6	18
K_{47}	243	2	9	6	27

Table B.37: Data for example 3.

p	$\theta_{K_p/\mathbb{Q},R_p}(0)$
229	$[-9, 24, -3, 12, 21, -30, 15, 6, 18, -18, -6, -15, 30, -21, -12, 3, -24, 9]/19$
7	$[12, -3, -18, 3, -3, 18, 3, -21, 9, -9, -3, -15, -12, 15, 12, -12, 21, 3]$
3389	$[15, -3, -9, -9, -12, 3, -12, 12, 6, -6, -15, 12, 9, 3, 3, -3, 9, -3]$
61	$[36, -36, -9, -45, 9, 45]$
5	$[-72, 18, -90, 90, 72, -18]$
47	$[-27, 27, 27, 54, -27, -54]$

- In K_{229} one has $\gamma(w_{K_{229}} \cdot \theta_{K_{229}/\mathbb{Q},R_{229}}(0)) = 6 = 2 \cdot 3$, thus

$$\frac{w_{K_{229}} \cdot \theta_{K_{229}/\mathbb{Q},R_{229}}(0)}{n_{229}} \in \mathbb{Z}[\Gamma_{229}].$$

  From the table below, we see that we always have

Table B.38: The Stickelberger element for the ramified prime 229.

t	$\gamma(\theta_{K_{229}/\mathbb{Q},R_{229},\{t\}}(0))$
3	3
11	3
13	3
17	3
23	3
29	3

$$\frac{\theta_{K_{229}/\mathbb{Q},R_{229},\{t\}}(0)}{n_{229}} \in \mathbb{Z}[\Gamma_{229}].$$

Moreover, since $h_{K_{229}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 229)$ is true by Theorem

5.24.

- We did not do the computations for the fields K_7 and K_{3389} since they are too lengthy.

- In K_{61}, one has $\gamma(w_{K_{61}} \cdot \theta_{K_{61}/\mathbb{Q},R_{61}}(0)) = 18 = 2 \cdot 9$, thus we have

$$\frac{w_{K_{61}} \cdot \theta_{K_{61}/\mathbb{Q},R_{61}}(0)}{n_{61}} \in \mathbb{Z}[\Gamma_{61}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We checked that

$$\mathfrak{p}_i^{\frac{\theta_{K_{61}/\mathbb{Q},R_{61}}(0)}{9}}$$

is principal and generated by an anti-unit (for $i = 1, 2$). Thus $BrSt(K/\mathbb{Q}, R_{61})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$2 \cdot \frac{\theta_{K_{661}/\mathbb{Q},R_{61}}(0)}{9},$$

and $w_{K_{61}} = 2$.

- In K_5, one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 36 = 4 \cdot 9$, thus we have

$$\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5],$$

but we already know this by Theorem 5.29. The class group is generated by the class of three primes $\mathfrak{p}_i$ ($i = 1, 2, 3$). We checked that

$$\mathfrak{p}_i^{\frac{\theta_{K_5/\mathbb{Q},R_5}(0)}{9}}$$

is principal and generated by an anti-unit (for $i = 1, 2, 3$). This implies that $BrSt(K/\mathbb{Q}, R_5)$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$2 \cdot \frac{\theta_{K_5/\mathbb{Q},R_5}(0)}{9},$$

and $w_{K_5} = 2$.

- In K_{47}, one has $\gamma(w_{K_{47}} \cdot \theta_{K_{47}/\mathbb{Q},R_{47}}(0)) = 54 = 6 \cdot 9$, thus we have

$$\frac{w_{K_{47}} \cdot \theta_{K_{47}/\mathbb{Q},R_{47}}(0)}{n_{47}} \in \mathbb{Z}[\Gamma_{47}],$$

  but we already know this by Theorem 5.29. The class group is generated by the class of three primes $\mathfrak{p}_i$ $(i = 1, 2, 3)$. We checked that

$$\mathfrak{p}_i^{\frac{\theta_{K_{47}/\mathbb{Q},R_{47}}(0)}{9}}$$

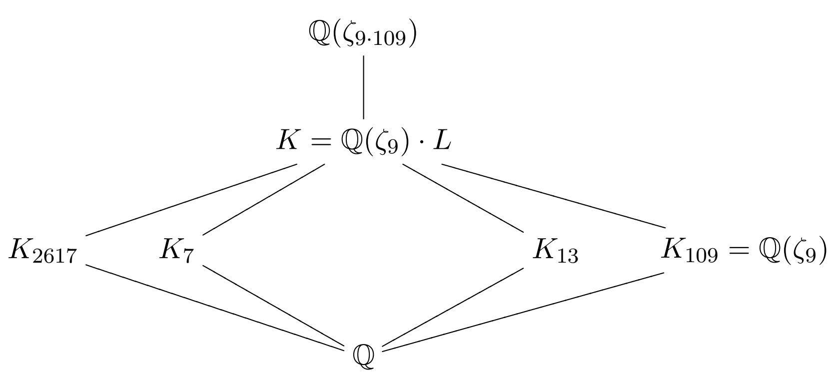
  is principal and generated by an anti-unit (for $i = 1, 2, 3$). This implies that $BrSt(K/\mathbb{Q}, R_{47})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$2 \cdot \frac{\theta_{K_{47}/\mathbb{Q},R_{47}}(0)}{9},$$

  and $w_{K_{47}} = 2$.

B.2.4 Example 4

Since $109 \equiv 1 \pmod 3$, let L be the unique subfield of $\mathbb{Q}(\zeta_{109})$ which is of degree 3 over $\mathbb{Q}$. Consider $K = \mathbb{Q}(\zeta_9) \cdot L$; it has only 2 ramified primes namely 3 and 109. It is an abelian extension of $\mathbb{Q}$ with Galois group $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. There are 4 minimal cocyclic subgroups and they are precisely the 4 subgroups of order 3. The set $S = \{\infty, 3, 109, 7, 13, 2617\}$ is a 1-cover and $S_{min} = \{109, 7, 13, 2617\}$. We have the following diagram of fields.



We computed the following data

Table B.39: Data for example 4.

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_{109}	1	18	3	6	$[1, -1, -2, 2, 1, -1]$	1
K_{2617}	3	6	3	6	$[-3, 3, 0, 0, -3, 3]$	3
K_7	171	6	3	6	$[9, -6, -15, -9, 15, 6]$	3
K_{13}	63	6	3	6	$[6, -3, -9, -6, 3, 9]$	3

- In K_{109}, one has $\gamma(w_{K_{109}} \cdot \theta_{K_{109}/\mathbb{Q},R_{109}}(0)) = 18 = 3 \cdot 6$, thus

$$\frac{w_{K_{109}} \cdot \theta_{K_{109}/\mathbb{Q},R_{109}}(0)}{n_{109}} \in \mathbb{Z}[\Gamma_{109}].$$

From the table below, we see that we always have

Table B.40: The Stickelberger element for the ramified prime 109.

t	$\gamma(\theta_{K_{109}/\mathbb{Q},R_{109},\{t\}}(0))$
5	3
11	3
17	$18 = 3 \cdot 6$
19	$18 = 3 \cdot 6$
23	3
29	3
31	3
37	$36 = 3 \cdot 12$

$$\frac{\theta_{K_{109}/\mathbb{Q},R_{109},\{t\}}(0)}{n_{109}} \in \mathbb{Z}[\Gamma_{109}].$$

Moreover, since $h_{K_{109}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 109)$ is true by Theorem 5.24.

- In K_{2617}, one has $\gamma(w_{K_{2617}} \cdot \theta_{K_{2617}/\mathbb{Q},R_{2617}}(0)) = 18 = 3 \cdot 6$, thus we have

$$\frac{w_{K_{2617}} \cdot \theta_{K_{2617}/\mathbb{Q},R_{2617}}(0)}{n_{2617}} \in \mathbb{Z}[\Gamma_{2617}],$$

but we already know this by Theorem 5.29. The class group is cyclic of order 3.

We chose a prime $\mathfrak{p}$ whose class generates the whole class group. We checked that

$$\mathfrak{p}^{\frac{\theta_{K_{2617}/\mathbb{Q},R_{2617}}(0)}{3}}$$

is principal and generated by an anti-unit. This implies that $BrSt(K/\mathbb{Q}, R_{2617})$ is true numerically. Note that in this case the abelian condition follows automatically since the element of the group ring we are interested in is

$$6 \cdot \frac{\theta_{K_{2617}/\mathbb{Q},R_{2617}}(0)}{3},$$

and $w_{K_{2617}} = 6$.

- In K_7, one has $\gamma(w_{K_7} \cdot \theta_{K_7/\mathbb{Q},R_7}(0)) = 18 = 3 \cdot 6$, thus we have

$$\frac{w_{K_7} \cdot \theta_{K_7/\mathbb{Q},R_7}(0)}{n_7} \in \mathbb{Z}[\Gamma_7],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. For both of them we checked that

$$\mathfrak{p}_i^{\frac{w_{K_7}\theta_{K_7/\mathbb{Q},R_7}(0)}{n_7}}$$

is a principal ideal generated by an anti-unit which satisfies the abelian condition. Hence, $BrSt(K/\mathbb{Q}, R_7)$ is true numerically.

- In K_{13}, one has $\gamma(w_{K_{13}} \cdot \theta_{K_{13}/\mathbb{Q},R_{13}}(0)) = 18 = 3 \cdot 6$, thus we have

$$\frac{w_{K_{13}} \cdot \theta_{K_{13}/\mathbb{Q},R_{13}}(0)}{n_{13}} \in \mathbb{Z}[\Gamma_{13}],$$

but we already know this by Theorem 5.29. The class group is generated by the class of two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. For both of them we checked that

$$\mathfrak{p}_i^{\frac{w_{K_{13}}\theta_{K_{13}/\mathbb{Q},R_{13}}(0)}{n_{13}}}$$

is a principal ideal generated by an anti-unit which satisfies the abelian condition. Hence, $BrSt(K/\mathbb{Q}, R_{13})$ is true numerically.

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