ON A GENERALIZATION OF THE RANK ONE RUBIN-STARK CONJECTURE

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Abstract. In this paper we study further the extended abelian rank one Stark conjecture contained in [4] and [5]. We formulate a *stronger question* (Question 4.2) which seems easier to investigate both theoretically and computationally. Question 4.2 includes a *generalization of the Brumer-Stark conjecture* on annihilation of class groups (see Question 4.7). We link it with a conjecture of Gross (contained in [6]), and in the process find some *new integrality properties* of the Stickelberger element (Theorem 4.30). Finally, we provide some *numerical examples* with base field $\mathbb{Q}$ for which Question 4.2, and thus the extended abelian rank one Stark conjecture, have an affirmative answer.

Contents

1. Introduction

The abelian rank one Stark conjecture is a statement concerning S-imprimitive L-functions of a finite abelian extension K/k of number fields where S is a finite set of places of k satisfying

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(1) $|S| \geq 2$,
(2) S contains the archimedean and the ramified places in K/k,
(3) S contains a place v_0 which splits completely in K/k.

These three conditions imply that the order of vanishing of the L-function $L_{K/k,S}(s,\chi)$ is at least one for all characters χ of $\mathrm{Gal}(K/k)$. On the other hand, it is not difficult to give examples of abelian extensions K/k and finite sets of places S satisfying conditions (1) and (2) above, but not (3), for which all the L-functions vanish with order of vanishing at least one as well (see Section 3). In his Ph.D. thesis under the supervision of Stark, Erickson formulated a conjecture in [5] and called it the extended abelian rank one Stark conjecture. A student of Popescu, Emmons, formulated a (S,T)-version of the conjecture for any order of vanishing ([4]) very much in the spirit of the original Rubin-Stark conjecture contained in [9].

In this paper, we study further the extended abelian rank one Stark conjecture, in both its S and (S,T)-version. In §2, we set up our notation and recall some results which we will use later on in this paper. The extended abelian rank one Stark conjecture of Erickson-Stark (S-version) and Emmons-Popescu ((S,T)-version) are stated in §3. In §4, we introduce a stronger question which implies the extended abelian rank one Stark conjecture and along the way obtain a potential generalization of the Brumer-Stark conjecture. We study its functoriality properties and we link it with a conjecture of Gross. We end this paper with §5, where we provide some numerical examples.

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2. Objects of study and previous results

2.1. L-functions and units. Let K/k be a finite abelian extension of number fields with galois group G. The symbol $S(K/k)$ denotes the collection of places of k which are either infinite or ramified in K/k. If R is any finite set of places of k, we let R_K be the collection of places of K lying above those in R. If S contains the infinite places of k, we denote the group of S_K-units of K by $E_{K,S}$. Its torsion subgroup μ_K is the group of roots of unity in K.

Given a finite set S of primes of k containing $S(K/k)$, the abelian rank one Stark conjecture predicts a link between the S_K-units $E_{K,S}$ and the value at $s=0$ of the S-imprimitive L-functions associated with these data. Given a character χ, we denote the corresponding L-function by $L_{K/k,S}(s,\chi)$. The order of vanishing of the S-imprimitive L-functions at $s=0$ is known thanks to the functional equation satisfied by these functions (for a proof, see [14], page 24, Proposition 3.4):

Theorem 2.1. *Let K/k be an abelian extension of number fields and let S be a finite set of primes of k containing the infinite ones. We have*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}} E_{K,S})^{\chi} = \begin{cases} |S|-1, & \text{if } \chi = \chi_1, \\ |\{v\in S \,|\, G_v \subseteq \mathrm{Ker}(\chi)\}|, & \text{if } \chi \neq \chi_1, \end{cases}$$

where G_v denotes the decomposition group associated to the place v in G.

The set of characters $\chi \in \widehat{G} = \mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{C}^{\times})$ whose S-imprimitive L-function have order of vanishing precisely one will be denoted by $\widehat{G}_{1,S}$.

It is convenient to formulate everything in an equivariant way and this is what we shall do throughout this paper. We denote the idempotents in the semi-simple algebra $\mathbb{C}[G]$ by $e_{\chi} = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$. If S is a finite set of primes of k containing $S(K/k)$, the S-equivariant L-function is defined to be

$$\theta_{K/k,S}(s) = \sum_{\chi\in\widehat{G}} L_{K/k,S}(s,\chi)e_{\bar{\chi}} = \sum_{\sigma\in G}\zeta_{K/k,S}(s,\sigma)\sigma^{-1},$$

where $\zeta_{K/k,S}(s,\sigma)$ denotes the S-partial zeta function associated to $\sigma \in G$.

We will also work with the (S,T)-imprimitive L-functions. If T is a finite set of places of k satisfying $S\cap T=\varnothing$, the (S,T)-equivariant L-function is defined to be

$$\theta_{K/k,S,T}(s)=\prod_{\mathfrak{p}\in T}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})^{1-s})\cdot\theta_{K/k,S}(s),$$

where $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism associated to $\mathfrak{p}$ in G and $\mathbb{N}$ denotes the absolute norm of an ideal. One has

$$\theta_{K/k,S,T}(s)=\sum_{\chi\in\widehat{G}}L_{K/k,S,T}(s,\chi)e_{\bar{\chi}}=\sum_{\sigma\in G}\zeta_{K/k,S,T}(s,\sigma)\sigma^{-1},$$

where the (S,T)-partial zeta function $\zeta_{K/k,S,T}(s,\sigma)$ is defined by the following equality in $\mathbb{C}[G]$:

$$\sum_{\sigma\in G}\zeta_{K/k,S,T}(s,\sigma)\sigma^{-1}=\prod_{\mathfrak{p}\in T}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})^{1-s})\sum_{\sigma\in G}\zeta_{K/k,S}(s,\sigma)\sigma^{-1}$$

We remark that these L-functions have the same order of vanishing at $s=0$ as the corresponding S-imprimitive L-functions with $T=\varnothing$. If $T\neq\varnothing$, then they are holomorphic and not only meromorphic as the S-imprimitive ones are. Those (S,T)-imprimitive L-functions are related to the (S_K,T_K)-units of K which we denote by $E_{K,S,T}$. Recall that this is the group of S_K-units in K which satisfy $\varepsilon\equiv 1 \bmod {}^{\times}\mathfrak{P}$, for all $\mathfrak{P}\in T_K$. Its torsion subgroup is denoted by $\mu_{K,T}$.

The Euler factors associated to primes in T are not arbitrary. In fact, one has the following important lemma whose proof is also contained in [14], page 82, Lemma 1.1.

Lemma 2.2. *Let K/k be an abelian extension of number fields with Galois group G, and let S be any finite set of primes of k containing $S(K/k)$ and the ones which divide w_K, the number of roots of unity of K. Then $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated over $\mathbb{Z}$ by the elements of the form $\sigma_{\mathfrak{p}}-\mathbb{N}(\mathfrak{p})$, where $\mathfrak{p}$ runs over all primes not in S.*

Lemma 2.2 implies the following one.

Lemma 2.3. *Let K/k be an abelian extension of number fields and let S be a finite set of primes of k containing $S(K/k)$. Then $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated over $\mathbb{Z}[G]$ by the elements of the form*

$$\delta_T=\prod_{\mathfrak{p}\in T}(1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

where T runs over all finite sets of primes in k satisfying $S\cap T=\varnothing$ and $\mu_{K,T}=1$.

We finish this section with a theorem which is due to Cassou-Noguès ([1]) and independently to Deligne and Ribet ([2]) .

Theorem 2.4. *Given any abelian extension of number fields K/k with Galois group G, a finite set S of primes of k containing $S(K/k)$ and a finite set T of primes satisfying $S\cap T=\varnothing$ and $\mu_{K,T}=1$, we have $\theta_{K/k,S,T}(0)\in\mathbb{Z}[G]$.*

Combined with Lemma 2.2, we also have $w_K\theta_{K/k,S}(0)\in\mathbb{Z}[G]$.

2.2. **Some Kummer theory.** Given a finite abelian extension K/k of number fields, the extensions of the form $K(\alpha^{1/w_K})/k$, where $\alpha\in K^\times$, appear recurrently in connection with the abelian rank one Stark conjecture so we recall some of their well-known properties. See [14], Proposition 1.2, page 83.

Theorem 2.5. *Suppose that K/k is a Galois extension of fields of characteristic 0 with Galois group Γ and suppose that $\mu_m\subseteq K^\times$. Let α be such that $\alpha^m=a\in K^\times$.*

1. *The extension $K(\alpha)/k$ is Galois if and only if there exist integers n_γ for $\gamma\in\Gamma$ such that $a^{\gamma-n_\gamma}\in(K^\times)^m$, for all $\gamma\in\Gamma$.*
2. *The extension $K(\alpha)/k$ is central if and only if given integers n_γ satisfying $\zeta^\gamma=\zeta^{n_\gamma}$ for all $\zeta\in\mu_m$, one has $a^{\gamma-n_\gamma}\in(K^\times)^m$, for all $\gamma\in\Gamma$*
3. *Suppose moreover that K/k is abelian. The extension $K(\alpha)/k$ is abelian if and only if given integers n_γ satisfying $\zeta^\gamma=\zeta^{n_\gamma}$ for all $\zeta\in\mu_m$ there exist $a_\gamma\in K^\times$ such that $a^{\gamma-n_\gamma}=a_\gamma^m$, for all $\gamma\in\Gamma$ and such that $a_{\gamma_1}^{\gamma_2-n_{\gamma_2}}=a_{\gamma_2}^{\gamma_1-n_{\gamma_1}}$, for all $\gamma_1,\gamma_2\in\Gamma$.*

2.3. **Hasse's principle for powers.**

Lemma 2.6. *Let K be a number field, m a positive integer dividing w_K, and $\mathfrak{p}$ a prime ideal relatively prime to m. If $\alpha \in K^\times$ is an element relatively prime to $\mathfrak{p}$, then*

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = 1 \text{ if and only if } \alpha \in (K_\mathfrak{p}^\times)^m,$$

where the symbol $\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m}$ denotes the m-th power residue symbol and $K_\mathfrak{p}$ is the localization of K at $\mathfrak{p}$.

Proof:

Consider the extension of local fields $K_\mathfrak{p}(\sqrt[m]{\alpha})/K_\mathfrak{p}$ where $\sqrt[m]{\alpha}$ is any m-th root of α. It is an unramified abelian extension; its Galois group is cyclic generated by the Frobenius automorphism. Now

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,m} = \sqrt[m]{\alpha}^{\sigma_\mathfrak{p}-1} = 1$$

if and only if $\sqrt[m]{\alpha}^{\sigma_\mathfrak{p}} = \sqrt[m]{\alpha}$. This last equality is equivalent to the statement $\alpha \in (K_\mathfrak{p}^\times)^m$.

Q.E.D.

This last lemma gives only local information. In order to be of any use globally, we need Hasse's principle for powers:

Theorem 2.7. *Let K be a number field and m be a positive integer. Let also S be a set of primes having density 1. Then the map $K^\times/(K^\times)^m \longrightarrow \prod_{v\in S} K_v^\times/(K_v^\times)^m$ is injective, except when the following three conditions are satisfied at the same time:*

1. *$m = 2^r m'$, with m' odd and $r \geq 3$,*
2. *$K(\mu_{2^r})/K$ is not cyclic,*
3. *All primes $\mathfrak{p} \in S$ dividing 2 splits in $K(\mu_{2^r})/K$.*

Proof:

For a proof, we refer to [7] page 530.

Q.E.D.

2.4. **Gross's conjecture.**

In this section we state a conjecture formulated by Gross in [6] which gives extra information about the Stark unit in the (S,T)-version of the abelian rank one Stark conjecture. We suppose that $|S| \geq 3$.

Let K/k be a finite abelian extension of number fields and suppose that L/k is another abelian extension satisfying $k \subseteq L \subseteq K$. Let S be a finite set of places of k containing $S(K/k)$, and let T be a finite set of finite places of k satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. Let $v \in S$ be a place which splits completely in L and fix a place $w \in S_L$ lying above v. We let $rec_w : L_w^\times \longrightarrow \mathrm{Gal}(K/L)$ be the local reciprocity map of local class field theory which we view as taking values in $\mathrm{Gal}(K/L)$ rather than in the Galois group of the corresponding extension of local fields. We note that since v splits in L/k, one has $L_w = k_v$. Throughout this paper, we let $|\cdot|_w$ denotes the absolute value corresponding to the place w normalized in such a way that the product formula holds.

Conjecture 2.8 (Stark-Gross)**.** *With the notation as above, there exists $\varepsilon_{S,T} \in E_{L,S,T}$ such that*

1. *$-\zeta'_{L/k,S,T}(0,\sigma) = \log|\varepsilon_{S,T}^\sigma|_w$, for all $\sigma \in \mathrm{Gal}(L/k)$,*
2. *$|\varepsilon_{S,T}|_{w'} = 1$ for all places w' of L not lying above v.*

Moreover,

$$rec_w(\varepsilon_{S,T}^{\gamma^{-1}}) = \prod_{\substack{\sigma\in\mathrm{Gal}(K/k)\\ \sigma|_L=\gamma}} \sigma^{\zeta_{K/k,S,T}(0,\sigma^{-1})},$$

for all $\gamma \in \mathrm{Gal}(L/k)$.

The first two predictions of this conjecture constitute the (S,T)-version of the abelian rank one Stark conjecture formulated in terms of partial zeta functions and involve the extension L/k only. The second part, involving the local reciprocity map, is due to Gross.

2.5. ***G*-modules.** Suppose that G is a finite abelian group and M is a G-module. If R is any ring, then we will abbreviate $R\otimes M$ by RM. We remark that $R[G]\otimes_{\mathbb{Z}[G]}M\simeq R\otimes_{\mathbb{Z}}M$, where the G-actions should be clear. The map $M\longrightarrow\mathbb{Q}M$ will be denoted by $m\mapsto\tilde{m}=1\otimes m$. The kernel of this map is precisely the $\mathbb{Z}$-torsion in M.

If we come back to the original setting at the beginning of this section where G is the Galois group of a finite abelian extension K/k of number fields and S is a finite set of places of k containing $S(K/k)$, then we define the idempotent $e_{1,S}=\sum_{\chi\in\widehat{G}_{1,S}}e_\chi$. A simple computation shows that $e_{1,S}\in\mathbb{Q}[G]$. Following [4], given any $\mathbb{Q}[G]$-module M, we can view it as a $\mathbb{C}[G]$-module after tensoring with $\mathbb{C}$, and we set $M_{1,S}=M\cdot e_{1,S}=\{m\in M\,|\,e_\chi m=0,\ \text{for all }\chi\notin\widehat{G}_{1,S}\}$.

3. The extended abelian rank one Stark conjecture

3.1. **Formulation of the conjecture.** Here is an example of a finite set S of primes of k containing $S(K/k)$ for which all the S-imprimitive L-functions have order of vanishing at least one, but such that it does not contain any prime which splits completely.

Take $k=\mathbb{Q}$, $K=\mathbb{Q}(\sqrt{-3},\sqrt{5})$ and $S=\{\infty,3,5,7,17\}$. Using the usual arithmetic of quadratic number fields, we find

$$G_3=G_5=\mathrm{Gal}(K/\mathbb{Q}),\ G_\infty=\mathrm{Gal}(K/\mathbb{Q}(\sqrt{5})),G_7=\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3})),\ \text{and } G_{17}=\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-15})).$$

The kernel of every non-trivial character of this Galois group contains at least one decomposition group; thus, all imprimitive L-functions associated to non-trivial characters have order of vanishing greater than or equal to one, by Theorem 2.1. Here, since $|S|\geq 2$, the order of vanishing of the S-imprimitive Riemann zeta function $L_{K/\mathbb{Q},S}(s,\chi_1)$ is at least one as well. Nevertheless, no prime in S splits completely in K.

Thus, it is natural to try to find a formula for $L'_{K/k,S}(0,\chi)$ which would generalize the usual abelian rank one Stark conjecture contained in [12] or [14]. This is what Stark and Erickson did in [5] and what we recall in this section.

Definition 3.1. Let K/k be an abelian extension of number fields with Galois group G and let Λ be any subset of $\widehat{G}$. Let S be any finite set of primes of k (perhaps not containing the ramified nor the archimedean primes). The set S is said to be a 1-*cover* for Λ if the following two conditions hold:

(1) For all non-trivial $\chi\in\Lambda$, there exists at least one prime $v\in S$ such that $G_v\subseteq\mathrm{Ker}(\chi)$,
(2) If the trivial character is in Λ, then $|S|\geq 2$.

In the case where $\Lambda=\widehat{G}$, we also say that S is a 1-cover for G (or for K/k) rather than for $\widehat{G}$.

Remark 1. *In the case where $S\supseteq S(K/k)$, and $\Lambda=\widehat{G}$, this means that $\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi)\geq 1$, for all $\chi\in\widehat{G}$, by Theorem 2.1.*

Definition 3.2. As usual, let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. The set S_{min} is defined as follows: It consists of all primes $v\in S$ for which there exists $\chi\in\widehat{G}_{1,S}$, $\chi\neq\chi_1$, such that $G_v\subseteq\mathrm{Ker}(\chi)$.

Remark 2. *The notation S_{min} is used because, as it is not hard to see, S_{min} is the minimal 1-cover of $\widehat{G}_{1,S}\smallsetminus\{\chi_1\}$. Note also that S_{min} is not necessarily itself a 1-cover of G.*

As an example, suppose that $S\supseteq S(K/k)$ and that S contains exactly one prime, say v, which splits completely. Then S is clearly a 1-cover of $\widehat{G}$ (if $|S|\geq 2$) and moreover, if $\widehat{G}_{1,S}\smallsetminus\{\chi_1\}\neq\varnothing$, then $S_{min}=\{v\}$.

If we come back to the example $K/\mathbb{Q}$ with $K=\mathbb{Q}(\sqrt{-3},\sqrt{5})$ and the 1-cover $S=\{\infty,3,5,7,17\}$, we see that $S_{min}=\{\infty,7,17\}$, since the corresponding decomposition groups of these three primes are precisely the three subgroups of order two of $\mathrm{Gal}(K/\mathbb{Q})$ and each such subgroup is precisely the kernel of a non-trivial character. In this case, S_{min} is itself a 1-cover of G. On the other hand, if we take $S=\{\infty,3,5,7,11,17\}$, then $S_{min}=\{7,17\}$, which is not a 1-cover of G.

We gave a biquadratic example, since it is the kind of extension of number fields of smallest degree for which there exist 1-covers not containing a prime which splits completely. Indeed, we have:

Proposition 3.3. *Suppose K/k is a finite cyclic extension of number fields with Galois group G. Then any 1-cover S for $\widehat{G}$ containing $S(K/k)$ has to contain at least one prime which splits completely in K/k. In the case where it contains exactly one split prime, say v, we have $S_{min} = \{v\}$.*

Proof:
This follows from the fact that any finite cyclic group has a non-trivial faithful character.

Q.E.D.

Definition 3.4. Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. For every $v \in S_{min}$, we fix a place $w = w(v) \in S_K$ lying above v. We define a $\mathbb{Z}[G]$-module morphism

$$R_{K/k,S} : E_{K,S} \longrightarrow \mathbb{C}[G], \text{ by the formula } u \mapsto -\sum_{v \in S_{min}} \frac{1}{|G_v|} \sum_{\sigma \in G} \log |u^{\sigma}|_w \cdot \sigma^{-1}.$$

We extend this morphism by $\mathbb{C}$-linearity to $\mathbb{C}E_{K,S}$ and we call it the *equivariant regulator*.

Notation 1. In the case where $S_{min} = \{v\}$, where v splits completely in K/k, we denote the regulator by $R_{K/k,w}$, where w is the fixed place lying above v.

The proof of the next proposition is contained in [4] (Proposition 3.2).

Proposition 3.5. *Let K/k be a finite abelian extension of number fields with Galois group G and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. The equivariant regulator induces an isomorphism of $\mathbb{C}[G]$-modules*

$$R_{K/k,S} : (\mathbb{C}E_{K,S})_{1,S} \xrightarrow{\simeq} (\mathbb{C}[G])_{1,S}.$$

If S is a 1-cover for $\widehat{G}$ containing $S(K/k)$, then $\theta'_{K/k,S}(0) \in (\mathbb{C}[G])_{1,S}$. We are now in the position to state the extended abelian rank one Stark conjecture (Conjecture 4.1 in [5]).

Conjecture 3.6 ($ESt(K/k,S)$)**.** *Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover of $\widehat{G}$ containing $S(K/k)$ and satisfying $S \neq S_{min}$. Then there exists $\varepsilon \in (\tilde{E}_{K,S})_{1,S}$ such that*

$$\theta'_{K/k,S}(0) = \frac{1}{w_K} \cdot R_{K/k,S}(\varepsilon). \tag{1}$$

Moreover, if $u \in E_{K,S}$ is such that $\tilde{u} = \varepsilon$, then the extension $K(u^{1/w_K})/k$ is abelian.

Let us make some remarks.

(1) In the case where $S_{min} = \{v\}$ and v is a prime which splits completely, the extended abelian rank one Stark conjecture reduces to the standard abelian rank one Stark conjecture denoted by $St(K/k,S,v)$ (or $St(K/k,S,v,w)$ if we want to specify the choice of the place w of K lying above v). In particular, this new conjecture does not say anything new in the case of cyclic extensions.
(2) If S is a 1-cover and $|S| = 2$, it can be shown that S contains a place which splits completely, see [4]. Therefore, we can avoid this case, since it is contained in the usual abelian rank one Stark conjecture.
(3) A unit satisfying equation (1) is also called a Stark unit as in the standard conjecture.
(4) This conjecture can be reinterpreted in terms of partial zeta functions. A simple computation shows that formula (1) is equivalent to

$$-w_K \zeta'_{K/k,S}(0,\tau) = \sum_{v \in S_{min}} \frac{1}{|G_v|} \log |\varepsilon^{\tau}|_w,$$

for all $\tau \in G$.

(5) The uniqueness of ε does not follow from Kronecker's theorem as in the usual abelian rank one Stark conjecture, since one cannot isolate its various absolute values. The uniqueness rather follows from Proposition 3.5.

Following [4], we introduce the following definition.

Definition 3.7. Let K/k be an abelian extension of number fields, S a 1-cover for $\widehat{G}$ containing $S(K/k)$. The *evaluator* $\varepsilon_{K/k,S}$ is defined to be the unique element in $(\mathbb{C}E_{K,S})_{1,S}$ satisfying

$$\theta'_{K/k,S}(0) = \frac{1}{w_K} \cdot R_{K/k,S}(\varepsilon_{K/k,S}).$$

With this definition, the extended abelian rank one Stark conjecture amounts to showing that $\varepsilon_{K/k,S} \in \tilde{E}_{K,S}$ (and the abelian condition).

So far, all known cases of this conjecture use the following proposition (which is a reformulation of the discussion contained in Section 5 of [5]).

Proposition 3.8. *Let K/k be a finite abelian extension of number fields and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. Suppose also that $|S| \geq 3$. For each $v \in S_{min}$ choose $w = w(v) \in S_K$ and $w' = w'(v) \in S_{K^{G_v}}$ between v and w, then*

$$\varepsilon_{K/k,S} = \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S}, \tag{2}$$

where $\varepsilon_{K^{G_v}/k,S} \in (\mathbb{C}E_{K^{G_v},S})_{1,S}$ satisfies

$$\theta'_{K^{G_v}/k,S}(0) = \frac{1}{w_{K^{G_v}}} \cdot R_{K^{G_v}/k,w'}(\varepsilon_{K^{G_v}/k,S}).$$

Proof:

Given $\chi \in \widehat{G}$, one has

$$e_\chi \cdot \varepsilon_{K^{G_v}/k,S} = \begin{cases} 0 & \text{if } \chi|_{G_v} \neq 1, \\ e_{\widetilde{\chi}} \cdot \varepsilon_{K^{G_v}/k,S} & \text{if } \chi|_{G_v} = 1, \end{cases} \tag{3}$$

where $\widetilde{\chi}$ takes the same values as χ, but is viewed as a character of G/G_v. Indeed, let $\sigma_1, \dots, \sigma_s$ be coset representatives for G/G_v, then

$$\begin{aligned} e_\chi \cdot \varepsilon_{K^{G_v}/k,S} &= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}\varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{h \in G_v} \chi(\sigma_i h)\sigma_i^{-1}h^{-1}\varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \sum_{h \in G_v} \chi(\sigma_i h)\sigma_i^{-1}\varepsilon_{K^{G_v}/k,S} \\ &= \frac{1}{|G|} \sum_{i=1}^{s} \chi(\sigma_i)\sigma_i^{-1} \left(\sum_{h \in G_v} \chi(h) \right) \varepsilon_{K^{G_v}/k,S} \end{aligned}$$

Noting that the quantity in parenthesis is $|G_v|$ or 0 depending on whether or not $G_v \subseteq \mathrm{Ker}(\chi)$, we conclude the validity of formula (3).

We prove formula (2) one character at the time. Since $|S| \geq 3$, both sides are equal to 0 when multiplied by e_{χ_1}. Let $\chi \in \widehat{G}$ be a non-trivial character and suppose first that $\chi \notin \widehat{G}_{1,S}$, then $e_\chi \cdot \varepsilon_{K/k,S} = 0$, since $\varepsilon_{K/k,S} \in (\mathbb{C}E_{K,S})_{1,S}$. On the right-hand side we have

$$e_\chi \cdot \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S} = e_{\widetilde{\chi}} \cdot \sum_{\chi|_{G_v}=1} \frac{w_K}{w_{K^{G_v}}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S}$$

by formula (3). But $e_{\widetilde{\chi}} \cdot \varepsilon_{K^{G_v}/k,S} = 0$ as well since the associated L-function also has a zero or order at least two.

Suppose now that $\chi \in \widehat{G}_{1,S}$, then we have on one hand $R_{K/k,S}(e_\chi \cdot \varepsilon_{K/k,S}) = w_K \cdot L'_S(0,\bar{\chi})$. If we multiply the right-hand side with e_χ, we get

$$\frac{w_K}{w_{K^{G_{v_0}}}} \frac{1}{|G_{v_0}|} \cdot e_\chi \cdot \varepsilon_{K^{G_{v_0}}/k,S},$$

where v_0 is the unique place in S such that $G_{v_0} \subseteq \mathrm{Ker}(\chi)$. Now, we apply $R_{K/k,S}$ to this last expression. For $v \in S_{min}$, $u \in E_{K^{G_v},S}$ and $\sigma_1, \ldots, \sigma_s$ coset representatives of G/G_{v_0}, we have

$$\begin{aligned} e_\chi \frac{1}{|G_v|} \sum_{\sigma \in G} \log|u^\sigma|_w \cdot \sigma^{-1} &= e_\chi \frac{1}{|G_v|} \sum_{i=1}^{s} \sum_{h \in G_{v_0}} \log|u^{\sigma_i h}|_w (\sigma_i h)^{-1} \\ &= e_\chi \frac{1}{|G_v|} \sum_{i=1}^{s} \left(\sum_{h \in G_{v_0}} h \right) \log|u^{\sigma_i h}|_w \sigma_i^{-1}. \end{aligned}$$

Noting that $e_\chi \sum_{h \in G_{v_0}} h = 0$ if $G_{v_0} \not\subseteq \mathrm{Ker}(\chi)$ and using the inflation property of Artin L-functions, we conclude that

$$R_{K/k,S}\left(\frac{w_K}{w_{K^{G_{v_0}}}} \frac{1}{|G_{v_0}|} \cdot e_\chi \cdot \varepsilon_{K^{G_{v_0}}/k,S} \right) = w_K \cdot \theta'_{K/k,S}(0) \cdot e_\chi = w_K \cdot L'_S(0, \overline{\chi}).$$

Q.E.D.

As explained in [5], this proposition can be used as follows. Suppose that the usual abelian rank one Stark conjecture is known for subextensions of K/k. Let $\varepsilon_v \in E_{K^{G_v},S}$ be a Stark unit for $St(K^{G_v}/k, S, v, w')$, that is $\tilde{\varepsilon}_v = \varepsilon_{K^{G_v}/k,S}$. Suppose we can show

(1) There exists $\eta_v \in K^\times$ such that $\eta_v^{|G_v|} = \varepsilon_v$.
(2) The extension $K(\eta_v^{\frac{1}{w_v}})/k$ is abelian, where we abbreviate $w_v = w_{K^{G_v}}$.

Then we would have

$$\varepsilon_{K/k,S} = \sum_{v \in S_{min}} \frac{w_K}{w_{K^{G_v}}} \tilde{\eta}_v \in \tilde{E}_{K,S},$$

and if $\varepsilon \in E_{K,S}$ is such that $\tilde{\varepsilon} = \varepsilon_{K/k,S}$, the extension $K(\varepsilon^{\frac{1}{w_K}})/k$ would be abelian since it would be contained in the compositum of the fields $K(\eta_v^{\frac{1}{w_v}})$ as v runs over all places in S_{min}.

At times, this method allows one to prove the conjecture if the usual abelian rank one Stark conjecture is already known for intermediate extensions of K/k.

This conjecture satisfies the usual functoriality properties as shown in [4] or [5]. See also Section 4.4 below.

Proposition 3.9. *If $S \subseteq S'$ then $ESt(K/k,S)$ implies $ESt(K/k,S')$. If $k \subseteq K' \subseteq K$ then $ESt(K/k,S)$ implies $ESt(K'/k,S)$.*

3.2. **The (S,T)-version of the extended conjecture.** In this section, we state the (S,T)-version of the extended abelian rank one Stark conjecture contained in [4].

Definition 3.10. Let K/k be an abelian extension of number fields and S a 1-cover for $\widehat{G}$ containing $S(K/k)$. Let also T be another finite set of primes of k satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. The (S,T)-*evaluator* $\varepsilon_{K/k,S,T}$ is defined to be the unique element in $(\mathbb{C}E_{K,S,T})_{1,S}$ satisfying

$$\theta'_{K/k,S,T}(0) = R_{K/k,S}(\varepsilon_{K/k,S,T}).$$

Conjecture 3.11 ($ESt(K/k,S,T)$)**.** *Let K/k be an abelian extension of number fields and S a 1-cover for $\widehat{G}$ containing $S(K/k)$ such that $S \neq S_{min}$. Let also T be another finite set of primes of k satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. Then $\varepsilon_{K/k,S,T} \in \tilde{E}_{K,S,T}$.*

The counterpart of Proposition 3.8 is given by the following proposition whose proof is contained in [4] for any order of vanishing situation.

Proposition 3.12. *Let K/k be a finite abelian extension of number fields and let S be a 1-cover for $\widehat{G}$ containing $S(K/k)$. Suppose also that $|S| \geq 3$. Let T be another finite set of primes satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$. For each $v \in S_{min}$ choose $w = w(v) \in S_K$ and $w'(v) \in S_{K^{G_v}}$ between v and w, then*

$$\varepsilon_{K/k,S,T} = \sum_{v \in S_{min}} \frac{1}{|G_v|} \varepsilon_{K^{G_v}/k,S,T},$$

where $\varepsilon_{K^{G_v}/k,S,T} \in (\mathbb{C}E_{K^{G_v},S,T})_{1,S}$ satisfies

$$\theta'_{K^{G_v}/k,S,T}(0) = R_{K^{G_v}/k,w'}(\varepsilon_{K^{G_v}/k,S,T}).$$

3.3. **Minimal cocyclic subgroups and 1-covers.** There is a notion which is useful in order to think about 1-covers: It is the notion of minimal cocyclic subgroups. Among other things, we use this notion to provide numerical examples at the end of this paper.

Definition 3.13. Let G be a finite abelian group. A subgroup H is called *cocyclic* if G/H is a cyclic group. Moreover, if H is cocyclic and if $K \subsetneq H$ implies that G/K is not cyclic, then we say that H is a *minimal cocyclic subgroup.*

This notion shows up because of the following reason. Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover of $\widehat{G}$. Let χ be a non-trivial character of G, and suppose that it does not have order two. We clearly have $\mathrm{Ker}(\chi) \subseteq \mathrm{Ker}(\chi^2)$; thus,

$$\mathrm{ord}_{s=0}(L_{K/k,S}(s,\chi)) \leq \mathrm{ord}_{s=0}(L_{K/k,S}(s,\chi^2)).$$

Lemma 3.14. *The cocyclic subgroups are precisely the kernels of the characters of G.*

Proof:

Let $H = \mathrm{Ker}(\chi)$ be the kernel of some character χ. Then χ can be viewed as a faithful character of G/H, and since the only finite subgroups of $\mathbb{C}^\times$ are cyclic, we conclude that G/H is a cyclic group.

Conversely, if H is a cocyclic subgroup of G, then G/H is a finite cyclic group, and for any such group, there exists a faithful character, say χ. Now, its inflation to G will have kernel precisely equal to H.

Q.E.D.

Theorem 3.15. *Let K/k be an abelian extension of number fields with Galois group G. Let $H_1, \ldots, H_m$ be the minimal cocyclic subgroups of G. A finite set S of primes is a 1-cover of $\widehat{G}$ if and only if for all $i = 1, \ldots, m$ we have*

$$G_v \subseteq H_i,$$

for some $v \in S$ (depending on i) and also $|S| \geq 2$.

Proof:

This follows immediately from Definition 3.1, Lemma 3.14, and the fact that any cocyclic subgroup contains a minimal cocyclic subgroup.

Q.E.D.

Theorem 3.16. *Let K/k be a finite abelian extension of number fields with Galois group G and suppose $G \neq 1$. Let $H_1, \ldots, H_m$ be the minimal cocyclic subgroups of G and let S be a 1-cover of $\widehat{G}$, then S_{min} consists precisely of the primes $v \in S$ such that there exists a H_i satisfying:*

(1) $G_v \subseteq H_i$,
(2) $G_{v'} \nsubseteq H_i$, *for all $v' \in S$, $v' \neq v$.*

Proof:

Let $\chi \in \widehat{G}_{1,S} \smallsetminus \{\chi_1\}$ and let v be the unique place in S satisfying $G_v \subseteq \mathrm{Ker}(\chi)$. We want to show first that there exists a minimal cocyclic subgroup H such that $G_v \subseteq H$. Let H be a minimal cocyclic subgroup contained in $\mathrm{Ker}(\chi)$. Since S is a 1-cover, there exists, by Theorem 3.15, $v' \in S$ such that $G_{v'} \subseteq H$. If $v' \neq v$, then we would have $\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) \geq 2$, and this would be a contradiction. Therefore, $v' = v$ and $G_v \subseteq H$. If there were another $v' \in S$ such that $G_{v'} \subseteq H$, it would be again a contradiction with the fact that $\chi \in \widehat{G}_{1,S}$.

Conversely, let $v \in S$ and suppose that there exists a minimal cocyclic subgroup H such that $G_v \subseteq H$ and $G_{v'} \nsubseteq H$ for all $v' \in S$, $v' \neq v$. We want to show that $v \in S_{min}$. By Lemma 3.14, we know that there exists a character χ such that $H = \mathrm{Ker}(\chi)$. If $\chi = \chi_1$, then G itself would be a minimal cocyclic subgroup, and this would imply that $G = 1$ which is excluded. Hence, $\chi \neq \chi_1$. If we show that $\chi \in \widehat{G}_{1,S}$, then we would be done. But this is precisely what the second condition means.

Q.E.D.

This notion is useful in order to compute explicit examples of 1-covers for a given abelian extension of number fields with Galois group G. One starts by finding the minimal cocyclic subgroups of G. Then one checks whether or not the decomposition groups associated to ramified and archimedean primes are contained in some of these minimal cocyclic subgroups which leads to the following definition.

Definition 3.17. Let K/k be an abelian extension of number fields with Galois group G and let S be a 1-cover for G. The minimal cocyclic subgroups containing a decomposition group G_v for some finite ramified or infinite place v in S will be called *distinguished*. If we want to emphasize the G_v contained in the minimal cocyclic subgroup H, we shall say that H is *v-distinguished*. Moreover, the collection of v-distinguished minimal cocyclic subgroups for finite ramified places v (resp. infinite places v) will be referred to as the *finite distinguished* minimal cocyclic subgroups (resp. *infinite distinguished*).

Finally, one fills in S by adding finite unramified primes such that their Frobenius automorphisms are contained in the minimal cocyclic subgroups which are not distinguished.

Since an example is worth a thousand words, let us look again at the biquadratic extension $K = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$. We denote its Galois group over $\mathbb{Q}$ by G. The cocyclic subgroups are

$$\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3})), \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5})), \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-15})), \text{ and } G.$$

The minimal cocyclic subgroups are the three subgroups of order two. There are two ramified primes which are 3 and 5 and their decomposition groups are $G_3 = G_5 = G$. Moreover, we have $G_\infty = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5}))$, and thus this last subgroup of order two is a distinguished minimal cocyclic subgroup. It is the only one. According to our procedure, we have to find finite unramified primes whose Frobenius automorphisms are contained in the non-distinguished minimal cocyclic subgroups. We can choose 7 and 17 for instance, and this is why $S = \{\infty, 3, 5, 7, 17\}$ is a 1-cover for G.

Because of the usefulness of this notion, we shall study it a little bit more.

Definition 3.18. Given a finite abelian group G, we let $n_d(G)$ (or n_d when G is fixed) denote the number of cyclic subgroups of order d. Moreover, we let n_G denote the number of cyclic subgroups of G. Hence

$$n_G = \sum_{d \mid n} n_d(G).$$

We also let n'_d (resp. n'_G) be the number of minimal cocyclic subgroups of order d (resp. of G).

Proposition 3.19. *Let G be a finite abelian group. The number of cocyclic subgroups of cardinality d is equal to $n_{|G|/d}(G)$.*

Proof:

We make use of the perfect pairing $\widehat{G} \times G \longrightarrow \mathbb{C}^\times$ coming from duality theory for finite abelian groups. The finite cocyclic subgroups of cardinality d of G correspond to finite cyclic subgroups of cardinality $|G|/d$ of $\widehat{G}$. Indeed, if H is a finite cocyclic subgroup of cardinality d, then $H^\perp \simeq \widehat{G/H} \simeq G/H$; thus, we conclude that $H^\perp$ is cyclic of cardinality $|G|/d$.

Conversely, if C is a cyclic subgroup of cardinality $|G|/d$ of $\widehat{G}$, then let χ be a generator of C. We claim that $C^{\perp} = \mathrm{Ker}(\chi)$. The inclusion $C^{\perp} \subseteq \mathrm{Ker}(\chi)$ is obvious from the definition. If $g \in \mathrm{Ker}(\chi)$, let $\psi \in C$. There exists $i \in \mathbb{Z}$ such that $\psi = \chi^i$ and thus $\psi(g) = 1$ as well. We conclude that $C^{\perp} = \mathrm{Ker}(\chi)$. Therefore, $C^{\perp}$ is cocyclic of cardinality d.

Q.E.D.

By duality theory for finite abelian groups, the number of minimal cocyclic subgroups is the same as the number of maximal cyclic subgroups.

The proofs of the next four results are elementary and left to the reader.

Theorem 3.20. *We have the following formula for any finite abelian group G:*

$$|G| = \sum_{d \mid |G|} n_d(G) \cdot \varphi(d),$$

where φ is the Euler φ-function.

Theorem 3.21. *Let G_1 and G_2 be two finite abelian groups and suppose that $(|G_1|, |G_2|) = 1$. For any subgroup H of $G_1 \times G_2$, there exist subgroups H_i of G_i such that*

$$H = H_1 \times H_2.$$

Corollary 3.22. *Let G_1 and G_2 be two finite abelian groups. If $G = G_1 \times G_2$, where $(|G_1|, |G_2|) = 1$, every cyclic subgroup of G is of the form $H_1 \times H_2$ where H_i is a cyclic subgroup of G_i. Thus*

$$n_G = n_{G_1} \cdot n_{G_2}.$$

We also have that every minimal cocyclic subgroup of G is of the form $H_1 \times H_2$, where H_i is a minimal cocyclic subgroup of G_i; therefore,

$$n'_G = n'_{G_1} \cdot n'_{G_2}.$$

Theorem 3.23. *Suppose that $G \simeq (\mathbb{Z}/p\mathbb{Z})^n$, then*

$$n'_G = n_p(G) = 1 + p + \ldots + p^{n-1} = \frac{p^n - 1}{p - 1}.$$

4. Investigation of the extended abelian rank one Stark conjecture

As we explained previously, the formulae of Proposition 3.8 and 3.12 are sometimes useful in order to show that Conjecture 3.6 or 3.11 hold true. The best thing one could hope for is that either one of the conjecture would be provable one place $v \in S_{min}$ at the time. This will lead us to formulate a stronger question than the original one. Before we carry on with this task, here are some observations.

In order to detect where exactly the various Stark units are powers, it might be useful to consider the following map. If K/L is a finite extension of number fields and n any positive integer, let $\varphi : L^{\times}/(L^{\times})^n \longrightarrow K^{\times}/(K^{\times})^n$ be induced by the natural inclusion $L^{\times} \to K^{\times}$.

Proposition 4.1. *Let K/L be a Galois extension of number fields and n a positive integer satisfying $(n, w_K) = 1$. Then the group homomorphism $\varphi : L^{\times}/(L^{\times})^n \longrightarrow K^{\times}/(K^{\times})^n$ is injective.*

Proof:

Let $x \in L^{\times}$ and suppose $x = y^n$ for some $y \in K^{\times}$. Let $\sigma \in \mathrm{Gal}(K/L)$ and consider $y^{\sigma-1}$. If we raise it to the power n, we get $(y^{\sigma-1})^n = x^{\sigma-1} = 1$. In other words, $y^{\sigma-1} \in \mu_n \cap K^{\times}$. But since $(n, w_K) = 1$, we get $\mu_n \cap K^{\times} = \{1\}$. Therefore, $y^{\sigma-1} = 1$ for all $\sigma \in \mathrm{Gal}(K/L)$ and we conclude that $y \in L^{\times}$.

Q.E.D.

Suppose we are in the following situation where we work with the S-formulation of the conjecture (Conjecture 3.6). Let K/k be a finite abelian extension of number fields and S a 1-cover. Let $\mathfrak{p} \in S_{min}$ be a finite prime, $n = |G_{\mathfrak{p}}|$, and $L = K^{G_{\mathfrak{p}}}$. Let ε be a Stark unit for $St(L/k, S, \mathfrak{p})$, and suppose also it is an n-th power in K, that is, there exists $\eta \in K^{\times}$ such that $\eta^n = \varepsilon$. Let us make the further hypothesis that $(n, w_K) = 1$. In this case, the last proposition shows that $\eta \in L^{\times}$.

Here is a remark concerning the abelian condition. Assume we have a tower of fields $k \subseteq L \subseteq K$ such that K/k is abelian and let $x \in L^{\times}$. Then the extension $K(x^{\frac{1}{w_L}})/k$ is abelian if and only if $L(x^{\frac{1}{w_L}})/k$ is abelian. This is clear since $K(x^{1/w_L})$ is the compositum $K \cdot L(x^{1/w_L})$.

If we introduce a finite set of finite primes T satisfying $S \cap T = \varnothing$ and $\mu_{K,T} = 1$, then the map $E_{L,S,T}/E^n_{L,S,T} \longrightarrow E_{K,S,T}/E^n_{K,S,T}$ is always injective for any choice of n whenever K/L is Galois by an argument similar to the one given in the proof of Proposition 4.1.

4.1. A stronger question, the S-version.

Notation 2. Throughout this section, we shall be dealing with an abelian extension K/k of number fields and a single prime $v \in S_{min}$, where S is a 1-cover containing $S(K/k)$. We shall denote $|G_v|$ by n and K^{G_v} by L. Moreover the quotient group $\Gamma_v = G/G_v$ will be referred to as Γ.

We will have to deal with sets S of places of k satisfying the following properties:

Hypothesis 4.1.
(1) *$S \supseteq S(K/k)$,*
(2) *S is a 1-cover for K/k,*
(3) *$S \neq S_{min}$.*

Question 4.2 ($St(K/k, S, v, w)$)**.** *Let K/k be an abelian extension of number fields and S a finite set of places satisfying Hypothesis 4.1. Fix $v \in S_{min}$ and $w \in S_L$ lying above v. Does there exists $\eta \in (\tilde{E}_{L,S})_{1,S}$ such that*

$$\theta'_{L/k,S}(0) = \frac{n}{w_L} \cdot R_{L/k,w}(\eta)? \tag{4}$$

Moreover, if $u \in E_{L,S}$ satisfies $\tilde{u} = \eta$, is the extension $L(u^{\frac{1}{w_L}})/k$ abelian?

We make some remarks:

(1) The uniqueness of η follows because we require $\eta \in (\tilde{E}_{L,S})_{1,S}$. If $|S| \geq 3$, we could replace this condition and require instead $|\eta|_{w'} = 1$ for all $w' \in S_L$ not lying above v.
(2) We shall denote this question by $St(K/k, S, v)$, where $v \in S_{min}$, or $St(K/k, S, v, w)$ if we want to emphasize the choice of $w \in S_L$. The truth of the question is independent of the choice of w (by an argument similar to the one given for the usual abelian rank one Stark conjecture), but the unit η does depend on the choice of w in a simple way.
(3) Moreover, in the case of an affirmative answer, the $\eta \in (\tilde{E}_{L,S})_{1,S}$ satisfying
$$\theta'_{L/k,S}(0) = \frac{n}{w_L} \cdot R_{L/k,w}(\eta),$$
will be called a Stark n-unit.
(4) If S contains a split prime v then either $S_{min} = \varnothing$ or $S_{min} = \{v\}$. In the latter case, $St(K/k, S, v)$ is just the usual abelian rank one Stark conjecture since $K = L$ and $n = 1$.
(5) A simple computation shows that equation (4) is equivalent to
$$-\frac{w_L}{n}\zeta'_{L/k,S}(0, \gamma) = \log|\eta^{\gamma}|_w,$$
for all $\gamma \in \Gamma$.
(6) Let $u_1, u_2 \in E_{L,S}$ be such that $\tilde{u}_1 = \eta$ and $\tilde{u}_2 = \varepsilon$ where ε is a Stark unit for the usual abelian rank one Stark conjecture $St(L/k, S, v, w)$, that is,
$$\theta'_{L/k,S}(0) = \frac{1}{w_L} \cdot R_{L/k,w}(\varepsilon).$$
We have $u_2 = \zeta u_1^n$ for some $\zeta \in \mu_L$, since $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-modules when restricted to the $()_{1,S}$ summand. In other words, one of the Stark units, namely $\zeta^{-1}u_2$, is an n-th power of u_1. If one intends to show that Question 4.2 has an affirmative answer by showing that a usual Stark unit in L is a power of some other unit, then the determination of the argument of a Stark unit becomes important.

Proposition 4.3. *Let K/k be an abelian extension of number fields and let S be a finite set of places satisfying Hypothesis 4.1. If $St(K/k,S,v)$ has an affirmative answer for all $v\in S_{min}$ then the extended abelian rank one Stark conjecture (Conjecture 3.6) is true.*

Proof:
This is a direct consequence of Proposition 3.8 and the remark following it.

Q.E.D.

4.1.1. *The case a finite prime.* Suppose we are in the setting of the question $St(K/k,S,v)$ where S is a 1-cover for K/k and $v\in S_{min}$. In this section, we derive some consequences in the case where v is a finite prime. Because of that, we switch of notation, and we denote v by $\mathfrak{p}$. Fix a prime $\mathfrak{P}$ of $L=K^{G_{\mathfrak{p}}}$ lying above $\mathfrak{p}$ and denote $S\smallsetminus\{\mathfrak{p}\}$ by $R_{\mathfrak{p}}$ or R when $\mathfrak{p}$ is fixed. Moreover, as before $n=n_{\mathfrak{p}}=|G_{\mathfrak{p}}|$, and $\Gamma=\Gamma_{\mathfrak{p}}=G/G_{\mathfrak{p}}$. Since

$$\theta'_{L/k,S}(s)=\frac{\sigma_{\mathfrak{p}}^{-1}}{\mathbb{N}(\mathfrak{p})^s}\cdot\log\mathbb{N}(\mathfrak{p})\cdot\theta_{L/k,R}(s)+\left(1-\frac{\sigma_{\mathfrak{p}}^{-1}}{\mathbb{N}(\mathfrak{p})^s}\right)\cdot\theta'_{L/k,R}(s),$$

and $\mathfrak{p}$ splits in L/k, one gets

$$\theta'_{L/k,S}(0)=\log\mathbb{N}(\mathfrak{p})\cdot\theta_{L/k,R}(0). \tag{5}$$

Now, if $St(K/k,S,\mathfrak{p},\mathfrak{P})$ has an affirmative answer then there exists $\eta\in(\tilde{E}_{L,S})_{1,S}$ such that

$$\begin{aligned}\theta'_{L/k,S}(0)&=\frac{n}{w_L}\cdot R_{L/k,\mathfrak{P}}(\eta)\\&=-\frac{n}{w_L}\cdot\sum_{\gamma\in\Gamma}\log|\eta|_{\mathfrak{P}^\gamma}\cdot\gamma\\&=\frac{n}{w_L}\cdot\sum_{\gamma\in\Gamma}\mathrm{ord}_{\mathfrak{P}^\gamma}(\eta)\cdot\log\mathbb{N}(\mathfrak{p})\cdot\gamma.\end{aligned}$$

Hence, combining this last line with equation (5), one gets

$$\theta_{L/k,R}(0)=\frac{n}{w_L}\cdot\sum_{\gamma\in\Gamma}\mathrm{ord}_{\mathfrak{P}^\gamma}(\eta)\cdot\gamma.$$

Therefore, one should have

$$\frac{w_L\cdot\theta_{L/k,R}(0)}{n}\in\mathbb{Z}[\Gamma]. \tag{6}$$

Moreover we get the following equality of ideals in L:

$$\mathfrak{P}^{\frac{w_L\theta_{L/k,R}(0)}{n}}=(\eta). \tag{7}$$

Combining (6) and (7), we arrive at the following conclusion.

Proposition 4.4. *Let K/k be an abelian extension of number fields, S a finite set of places satisfying Hypothesis 4.1, and suppose that $|S|\geq 3$. Let $\mathfrak{p}$ be a finite prime in S_{min}. With the same notation as above, if $St(K/k,S,\mathfrak{p},\mathfrak{P})$ has an affirmative answer with Stark n-unit $\eta=\tilde{u}$ then the following statements hold true:*

(1) $\frac{w_L\theta_{L/k,R}(0)}{n}\in\mathbb{Z}[\Gamma]$,
(2) $\mathfrak{P}^{\frac{w_L\theta_{L/k,R}(0)}{n}}=(u)$ *as ideals in* L,
(3) *The extension* $L(u^{\frac{1}{w_L}})/k$ *is abelian,*
(4) u *is an anti-unit.*

Conversely, if (1) *holds true and if there exists* $u\in L^\times$ *for which* (2),(3), *and* (4) *hold true then* $St(K/k,S,\mathfrak{p},\mathfrak{P})$ *has an affirmative answer with Stark* n*-unit* $\eta=\tilde{u}$.

Proof:
Just reverse the steps before the proposition.

Q.E.D.

Notation 3. From now on, we shall denote the group of anti-units in a number field K by K^0.

4.1.2. *An extension of Brumer-Stark.* In this section, not only do we assume that $\mathfrak{p} \in S_{min}$ is a finite prime, but we suppose that $\mathfrak{p}$ is unramified as well.

Definition 4.5. Let K/k be an abelian extension of number fields and let R be a finite set of primes containing $S(K/k)$. Let $\mathfrak{p} \notin R$ be any finite prime, necessarily unramified. The set R is called a $\mathfrak{p}$-1-cover if $S = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k, $\mathfrak{p} \in S_{min}$ and $S \neq S_{min}$.

Remark 3. *A* $\mathfrak{p}$-1-*cover is not necessarily a* 1-*cover.*

Definition 4.6. Let K/k be an abelian extension of number fields and R a finite set of places containing $S(K/k)$ which is a $\mathfrak{p}$-1-cover for K/k. We define $A = A_{R,L}$ to be the subgroup of Cl_L generated by the $[\mathfrak{Q}]$ where $\mathfrak{Q}$ is a prime ideal of L lying above a prime $\mathfrak{q} \notin R$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Moreover, we define $I_L^* = I_{L,R}^*$ to be the subgroup of I_L satisfying $I_L^*/P_L = A$.

Question 4.7 ($BrSt(K/k,R)$)**.** *With the notation as above, are the following statements true if* $|R| \geq 2$*?*

(1) $\frac{w_L \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,
(2) *For any* $\mathfrak{a} \in I_L^*$*, there exists* $\eta_R(\mathfrak{a}) \in L^\times$ *satisfying* $\mathfrak{a}^{\frac{w_L \cdot \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{a}))$ *as ideals in* L*,*
(3) $\eta_R(\mathfrak{a}) \in L^0$*,*
(4) *The extension* $L(\eta_R(\mathfrak{a})^{\frac{1}{w_L}})/k$ *is abelian.*

We make some remarks.

(1) We shall denote this question by $BrSt(K/k,R)$ where R is a given $\mathfrak{p}$-1-cover.
(2) Suppose that R is any finite set of places containing $S(K/k)$, and $\mathfrak{p} \notin R$ is a place which splits completely in K/k. The set $S = R \cup \{\mathfrak{p}\}$ is automatically a 1-cover. If $S_{min} \neq \varnothing$, then $S_{min} = \{\mathfrak{p}\}$ necessarily and we get back the original Brumer-Stark conjecture.

Lemma 4.8. *Let* R *be a* $\mathfrak{p}$-1-*cover for* K/k*. Let* $\mathfrak{q}$ *be any finite unramified prime such that*

(1) $\mathfrak{q} \notin R$*;*
(2) $\left(\frac{K/k}{\mathfrak{q}}\right) \in G_{\mathfrak{p}}$.

Then $S = R \cup \{\mathfrak{q}\}$ *is a* 1-*cover for* K/k*. Moreover* $\mathfrak{q} \in S_{min}$*.*

Proof:

Since R is a $\mathfrak{p}$-1-cover, we know that $S' = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k and $\mathfrak{p} \in S'_{min}$. The fact that S' is a 1-cover implies that every non-trivial character has at least one decomposition group in its kernel, and the fact that $\mathfrak{p} \in S'_{min}$ shows that there exists a non-trivial character $\chi_0 \in \widehat{G}_{1,S}$ such that $G_{\mathfrak{p}}$ is the unique decomposition group contained in $\mathrm{Ker}(\chi_0)$. If we replace S' by $S = R \cup \{\mathfrak{q}\}$, we see that it is still a 1-cover and $G_{\mathfrak{q}}$ is the unique decomposition group contained in $\mathrm{Ker}(\chi_0)$, that is $\mathfrak{q} \in S_{min}$.

Q.E.D.

Proposition 4.9. *Let* R *be a* $\mathfrak{p}$-1-*cover. Then* $BrSt(K/k,R)$ *has an affirmative answer if and only if* $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$ *has an affirmative answer for all finite primes* $\mathfrak{q} \notin R$ *satisfying*

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Proof:

Note first that by Lemma 4.8, it makes sense to talk about $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$. Suppose first that $BrSt(K/k, R)$ holds and let $\mathfrak{q} \notin R$ be a prime ideal of k satisfying the condition of the proposition. Let $\mathfrak{Q}$ be an ideal of $L = K^{G_{\mathfrak{p}}}$ lying above $\mathfrak{q}$. Note that $\mathfrak{Q} \in I_L^*$ and thus $BrSt(K/k, R)$ implies that there exists $\eta_R(\mathfrak{Q}) \in L^0$ satisfying

$$\mathfrak{Q}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{Q})),$$

and such that $L(\eta_R(\mathfrak{Q})^{\frac{1}{w_L}})/k$ is abelian. We can now conclude the desired result by Proposition 4.4.

Conversely, suppose that $St(K/k, R \cup \{\mathfrak{q}\}, \mathfrak{q})$ is true for all $\mathfrak{q} \notin R$ satisfying the condition above. Let $\mathfrak{a} \in I_L^*$, then $[\mathfrak{a}] \in A$. Now, by definition of A we have $[\mathfrak{a}] = C_1^{r_1} \cdot \ldots \cdot C_s^{r_s}$, where, for all $i = 1, \ldots, s$, there exists a prime $\mathfrak{Q}_i$ of L such that $[\mathfrak{Q}_i] = C_i$. Moreover, $\mathfrak{Q}_i$ lies above another prime $\mathfrak{q}_i$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}_i}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

There exists $\alpha \in L^\times$ such that

$$\mathfrak{a} = \alpha \cdot \mathfrak{Q}_1^{r_1} \cdot \ldots \cdot \mathfrak{Q}_s^{r_s}.$$

Now, $St(K/k, R \cup \{\mathfrak{q}_i\}, \mathfrak{q}_i, \mathfrak{Q}_i)$ implies that there exists $\eta_R(\mathfrak{Q}_i) \in L^0$ satisfying $\mathfrak{Q}_i^{\frac{w_L \theta_{L/k,R}(0)}{n}} = (\eta_R(\mathfrak{Q}_i))$ and such that $L(\eta_R(\mathfrak{Q}_i)^{\frac{1}{w_L}})/k$ is abelian. Therefore, we have

$$\mathfrak{a}^{\frac{w_L \theta_{L/k,R}(0)}{n}} = \alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}} \cdot \prod_{i=1}^{s} \eta_R(\mathfrak{Q}_i)^{r_i},$$

and we set

$$\eta_R(\mathfrak{a}) = \alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}} \cdot \prod_{i=1}^{s} \eta_R(\mathfrak{Q}_i)^{r_i}.$$

The algebraic number $\alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}}$ is an anti-unit by Lemma 4.10 below. We will show that it is w_L-abelian over k using Lemma 4.12 below. For each $\gamma \in \Gamma$, let $n_\gamma \in \mathbb{Z}$ be such that $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_L$. By Lemma 4.11 below, we get

$$\frac{(\gamma - n_\gamma) \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma],$$

for all $\gamma \in \Gamma$. Hence, we can apply Lemma 4.12 below, and we conclude that $\alpha^{\frac{w_L \theta_{L/k,R}(0)}{n}}$ is w_L-abelian over k.

Q.E.D.

Lemma 4.10. *Let K/k be a finite abelian extension of number fields with Galois group G and R a finite set of places containing $S(K/k)$ and satisfying $|R| \geq 2$. Given $v \in R$, we have $N_{G_v} \theta_{K/k,R}(0) = 0$ where $N_{G_v} = \sum_{\sigma \in G_v} \sigma \in \mathbb{Z}[G]$.*

Proof:

If χ_1 is the trivial character, then $e_{\chi_1} \cdot N_{G_v} \theta_{K/k,R}(0) = |G_v| \cdot \zeta_{k,R}(0) = 0$, since $|R| \geq 2$. If $\chi \neq \chi_1$ there are two possibilities: Either $G_v \subseteq \mathrm{Ker}(\chi)$ or not. In the former case, $e_\chi \cdot \theta_{K/k,R}(0) = 0$, and in the latter case $e_\chi \cdot N_{G_v} = 0$ by the orthogonality relations. We can then conclude the desired result.

Q.E.D.

Lemma 4.11. *Let K/k be an abelian extension of number fields with Galois group G. Suppose R is a finite set of places containing $S(K/k)$ which is a $\mathfrak{p}$-1-cover for K/k and let $S = R \cup \{\mathfrak{p}\}$. If $St(K/k, S, \mathfrak{p})$ has an affirmative answer, then*

$$\frac{\delta \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma],$$

for all $\delta \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$.

Proof:

Let $\mathfrak{P}$ be a prime ideal of L lying above $\mathfrak{p}$ and let $\eta \in L^\times$ be a Stark n-unit for $St(K/k, S, \mathfrak{p}, \mathfrak{P})$. Let I_L be the group of fractional ideals of L. For the purpose of this proof, we will think of I_L as being an additive free abelian group. We have

$$\frac{w_L \cdot \theta_{L/k,R}(0)}{n} \cdot \mathfrak{P} = (\eta). \tag{8}$$

Let λ be such that $\lambda^{w_L} = \eta$. The extension $L(\lambda)/k$ is abelian, since we assumed $St(K/k, S, \mathfrak{p}, \mathfrak{P})$ to have an affirmative answer. If $\delta \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$ and $\tilde{\delta} \in \mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(L(\lambda)/k)]}(\mu_{L(\lambda)})$ is a lift of δ, then $\lambda^{\tilde{\delta}} \in L^\times$. Indeed, given $\sigma \in \mathrm{Gal}(L(\lambda)/L)$, there exists $\zeta \in \mu_L$ such that $\lambda^\sigma = \zeta\lambda$. Hence

$$(\lambda^{\tilde{\delta}})^\sigma = (\lambda^\sigma)^{\tilde{\delta}} = (\zeta\lambda)^{\tilde{\delta}} = \lambda^{\tilde{\delta}}.$$

Equation (8) implies

$$\frac{\delta \cdot \theta_{L/k,R}(0)}{n} \cdot \mathfrak{P} = (\lambda^{\tilde{\delta}}),$$

an equality in $\mathbb{Q}I_L$. Since the right-hand side is in I_L, and $\mathfrak{p}$ splits completely in L/k, we conclude that

$$\frac{\delta \cdot \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma].$$

Q.E.D.

Lemma 4.12. *Let K/k be a finite abelian extension of number fields with Galois group G and R a finite set of places containing $S(K/k)$. For each $\sigma \in G$, choose $n_\sigma \in \mathbb{Z}$ such that $\zeta^\sigma = \zeta^{n_\sigma}$ for all $\zeta \in \mu_K$. Let n be any integer and suppose that*

1. $\frac{w_K \cdot \theta_{K/k,R}(0)}{n} \in \mathbb{Z}[G]$,
2. $\frac{(\sigma - n_\sigma)\cdot \theta_{K/k,R}(0)}{n} \in \mathbb{Z}[G]$.

Then given any $\alpha \in K^\times$, the algebraic number $\alpha^{\frac{w_K \cdot \theta_{K/k,R}(0)}{n}} \in K^\times$ is w_K-abelian over k.

Proof:

We use Theorem 2.5. Let $\beta = \alpha^{\frac{w_K \cdot \theta_{K/k,R}(0)}{n}} \in K^\times$ and set $\beta_\sigma = \alpha^{\frac{(\sigma - n_\sigma)\cdot \theta_{K/k,R}(0)}{n}}$. We clearly have $\beta^{\sigma - n_\sigma} = \beta_\sigma^{w_K}$ and $\beta_\sigma^{\tau - n_\tau} = \beta_\tau^{\sigma - n_\sigma}$ for all $\sigma, \tau \in G$.

Q.E.D.

4.2. **The (S,T)-version of the stronger question.** The purpose of this section is to formulate a (S,T)-version of Question 4.2. We have to introduce another finite set of finite primes T which satisfies the following properties.

Hypothesis 4.2.

1. $S \cap T = \varnothing$,
2. $\mu_{K,T} = 1$.

Question 4.13 ($St(K/k, S, T, v, w)$)**.** *Let K/k be an abelian extension of number fields, S a finite set of places of k satisfying Hypothesis 4.1, and T a finite set of finite places satisfying Hypothesis 4.2. Fix $v \in S_{min}$ and $w \in S_L$ lying above v. With the same notation as before does there exist $\eta \in (E_{L,S,T})_{1,S}$ such that*

$$\theta'_{L/k,S,T}(0) = n \cdot R_{L/k,w}(\eta)? \tag{9}$$

We make some remarks:

1. The uniqueness of η follows because we require $\eta \in (E_{L,S,T})_{1,S}$. If $|S| \geq 3$, we could replace this condition and require instead that $|\eta|_{w'} = 1$ for all $w' \in S_L$ not lying above v.
2. We shall denote this question by $St(K/k, S, T, v)$ where $v \in S_{min}$ or $St(K/k, S, T, v, w)$ if we want to specify the choice of $w \in S_L$. As before, the veracity of the question does not depend on the choice of w.

(3) Moreover, in the case of an affirmative answer, the $\eta \in (E_{L,S,T})_{1,S}$ satisfying
$$\theta'_{L/k,S,T}(0) = n \cdot R_{L/k,w}(\eta),$$
will be called a Stark n-unit.

(4) If S contains a split prime v then either $S_{min} = \varnothing$ or $S_{min} = \{v\}$. In the latter case, $St(K/k, S, T, v)$ is just the usual (S,T)-version of the abelian rank one Stark conjecture since $K = L$ and $n = 1$ in that case.

(5) The usual (S,T)-version of the abelian rank one of Stark conjecture for L/k predicts the existence of an $\varepsilon \in (E_{L,S,T})_{1,S}$, satisfying
$$\theta'_{L/k,S,T}(0) = R_{L/k,w}(\varepsilon).$$
Since $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-module when restricted to the $()_{1,S}$ summand, one should have $\varepsilon = \eta^n$. In other words, in the case of an affirmative answer and in the setting of our question, the Stark unit for $St(L/k, S, T, v)$ should in fact be a n-th power of some other unit which we call a Stark n-unit.

(6) Another way of looking at equation (9) is to say that $R^{-1}_{L/k,w}(\theta'_{L/k,S,T}(0)) \in (E_{L,S,T})^n_{1,S}$ which is a smaller subgroup then $(E_{L,S,T})_{1,S}$.

Proposition 4.14. *Let K/k be an abelian extension of number fields, and let S and T be finite sets of places satisfying Hypotheses 4.1 and 4.2. If $St(K/k, S, T, v)$ has an affirmative answer for all $v \in S_{min}$, then the (S,T)-version of the extended abelian rank one Stark conjecture (Conjecture 3.11) is true.*

Proof:

This is a direct consequence of Proposition 3.12

Q.E.D.

Proposition 4.15. *Let K/k be an abelian extension of number fields, and let S and T be finite sets of places satisfying Hypotheses 4.1 and 4.2. Assuming the usual abelian rank one Stark conjecture, if $S_{min} = \{v\}$ consists of a unique place, then $St(K/k, S, T, v)$ is equivalent to Conjecture 3.11.*

Proof:

This is a consequence of Proposition 3.12 and the fact that the map $E_{L,S,T}/E^n_{L,S,T} \longrightarrow E_{K,S,T}/E^n_{K,S,T}$ is injective, where $L = K^{G_v}$, and $n = |G_v|$.

Q.E.D.

It is simple to find examples where $|S_{min}| = 1$. In fact, if one enlarges S, then S_{min} stays the same or become smaller. Using this principle let us come back to the example $\mathbb{Q}(\sqrt{-3}, \sqrt{5})/\mathbb{Q}$. We saw that $S = \{\infty, 3, 5, 7, 17\}$ is a 1-cover such that $S_{min} = \{\infty, 7, 17\}$. If we look at $S = \{\infty, 3, 5, 7, 11, 17\}$ instead, we get $S_{min} = \{7, 17\}$ and at last if $S = \{\infty, 3, 5, 7, 11, 13, 17\}$, a simple computation shows that $S_{min} = \{17\}$.

We can also formulate a (S,T)-version of Proposition 4.4 and Question 4.7.

Proposition 4.16. *Let K/k be an abelian extension of number fields and S a finite set of places satisfying Hypothesis 4.1. Suppose also that $|S| \geq 3$. Let T be another finite set of finite primes satisfying Hypothesis* 4.2. *Let $\mathfrak{p}$ be a finite prime in S_{min} and set $R = S \smallsetminus \{\mathfrak{p}\}$. Then $St(K/k, S, T, \mathfrak{p}, \mathfrak{P})$ has an affirmative answer if and only if the following statements hold:*

(1) $\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) *There exists $\eta \in L^\times$ such that $\mathfrak{P}^{\frac{\theta_{L/k,R,T}(0)}{n}} = (\eta)$ as ideals in L,*

(3) $\eta \equiv 1 \mod {}^\times w$ *for all place $w \in T_L$,*

(4) *η is an anti-unit.*

We let $I_{L,T}$ be the subgroup of the group of fractional ideals of L consisting of ideals relatively prime to the primes in T_L. Also, let $P_{L,T}$ the subgroup of $I_{L,T}$ consisting of principal ideals generated by elements $\alpha \in L^\times$ of the form
$$\alpha \equiv 1 \mod {}^\times w,$$

for all $w \in T_L$. The T-class group of L is the quotient $Cl_T = I_{L,T}/P_{L,T}$.

Definition 4.17. Let K/k be an abelian extension of number fields, R a finite set of places containing $S(K/k)$ which is a $\mathfrak{p}$-1-cover for K/k. Let also T be a finite set of finite primes satisfying $(R \cup \{\mathfrak{p}\}) \cap T = \varnothing$ and $\mu_{K,T} = 1$. Define $A_T = A_{T,R,L}$ to be the subgroup of $Cl_{L,T}$ generated by the $[\mathfrak{Q}]$ where $\mathfrak{Q}$ is a prime ideal of L lying above a prime $\mathfrak{q} \notin R \cup T$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

Let also $I^*_{L,T}$ be the subgroup of $I_{L,T}$ satisfying $I^*_{L,T}/P_{L,T} = A_T$.

Question 4.18. *With the notation as above, are the following statements true?*

(1) $\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma]$,
(2) *For any* $\mathfrak{a} \in I^*_{L,T}$, *there exists* $\eta_{R,T}(\mathfrak{a}) \in L^\times$ *satisfying* $\mathfrak{a}^{\frac{\theta_{L/k,R,T}(0)}{n}} = (\eta_{R,T}(\mathfrak{a}))$,
(3) $\eta_{R,T}(\mathfrak{a}) \equiv 1 \mod {}^\times\mathfrak{P}$ *for all* $\mathfrak{P} \in T_L$,
(4) $\eta_{R,T}(\mathfrak{a}) \in L^0$.

We shall denote this conjecture by $BrSt(K/k, R, T)$ where R is a given $\mathfrak{p}$-1-cover.

4.3. Equivalence between the S-version and the (S,T)-version.

Theorem 4.19. *Let K/k be an abelian extension of number fields, S a finite set of primes satisfying Hypothesis 4.1, and choose $v \in S_{min}$. As usual let $L = K^{G_v}$. If $St(K/k, S, v, w)$ has an affirmative answer then $St(K/k, S, T, v, w)$ has an affirmative answer for all finite sets of finite primes T satisfying Hypothesis 4.2.*

More precisely, if η_S is a Stark n-unit for $St(K/k, S, v, w)$, then pick a w_L-th root λ of η_S. We know that $F = L(\lambda)$ is abelian over k. For each $\mathfrak{p} \in T$, fix a lift of $\left(\frac{L/k}{\mathfrak{p}}\right)$ to F and call it $\tilde{\sigma}_\mathfrak{p}$. Let

$$\alpha = \prod_{\mathfrak{p} \in T} \left(1 - \tilde{\sigma}_\mathfrak{p}^{-1} \mathbb{N}(\mathfrak{p})\right) \in \mathbb{Z}[\mathrm{Gal}(F/k)].$$

Then, $\eta_{S,T} = \lambda^\alpha$ is the Stark n-unit for $St(K/k, S, T, v, w)$.

Proof:
We first check that $\eta_{S,T} \in L^\times$. Let $\sigma \in \mathrm{Gal}(F/L)$. There exists $\zeta \in \mu_L$ such that $\lambda^\sigma = \zeta\lambda$. Thus

$$\eta_{S,T}^\sigma = (\lambda^\alpha)^\sigma = (\lambda^\sigma)^\alpha = (\zeta\lambda)^\alpha = \lambda^\alpha = \eta_{S,T}.$$

This last chain of equalities is true because F is abelian over k and $\zeta^\alpha = 1$. If $\pi : \mathbb{Z}[\mathrm{Gal}(F/k)] \longrightarrow \mathbb{Z}[\mathrm{Gal}(L/k)]$ denotes the usual restriction map, we have $\pi(\alpha) = \prod_{\mathfrak{p} \in T} \left(1 - \left(\frac{L/k}{\mathfrak{p}}\right)^{-1} \mathbb{N}(\mathfrak{p})\right)$. Setting $\delta_T = \pi(\alpha)$, we compute

$$\begin{aligned}
\theta'_{S,T}(0) &= \delta_T \theta'_S(0) \\
&= \frac{n \cdot \delta_T}{w_L} R_{L/k,w}(\eta_S) \\
&= \frac{n}{w_L} R_{L/k,w}(\eta_S^{\delta_T}) \\
&= \frac{n}{w_L} R_{L/k,w}((\lambda^\alpha)^{w_L}) \\
&= n \cdot R_{L/k,w}(\eta_{S,T}).
\end{aligned}$$

Remarking that $\eta_{S,T} \equiv 1 \mod {}^\times\mathfrak{P}$ for all $\mathfrak{P} \in T_L$, we conclude the desired result.

Q.E.D.

Theorem 4.20. *Let K/k be an abelian extension of number fields and let S be a finite set of primes satisfying Hypothesis 4.1. Let $v \in S_{min}$, $L = K^{G_v}$, $\Gamma = \mathrm{Gal}(L/k)$, $n = [K : L]$ and Ω be a finite set of finite primes such that $S \cap \Omega = \varnothing$. Suppose also that $\{1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p}) \,|\, \mathfrak{p} \in \Omega\}$ generates $\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$ over $\mathbb{Z}$. If $St(K/k, S, T, v, w)$ has an affirmative answer for all $T = \{\mathfrak{p}\}$ where $\mathfrak{p}$ runs over all primes in Ω, then $St(K/k, S, v, w)$ has an affirmative answer.*

More precisely, if $\eta_{S,\mathfrak{p}}$ is the Stark n-unit for $St(K/k, S, \{\mathfrak{p}\}, v, w)$ and

$$w_L = \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}}(1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

then a Stark unit for $St(K/k, S, v, w)$ is given by

$$\eta_S = \prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}}.$$

Proof:

For $\gamma \in \Gamma$, choose integers n_γ such that $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_L$. Since w_L and $\gamma - n_\gamma$ are in $\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$, one can write

$$w_L = \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}}(1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})), \text{ and } \gamma - n_\gamma = \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p},\gamma}(1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})),$$

for some $n_{\mathfrak{p}}, n_{\mathfrak{p},\gamma} \in \mathbb{Z}$. Then, set

$$\eta_S = \prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}}, \text{ and } \alpha_\gamma = \prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p},\gamma}}.$$

We have

$$\begin{aligned}
w_L \cdot \theta'_{L/k,S}(0) &= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot (1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p}))\theta'_{L/k,S}(0) \\
&= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot \theta'_{L/k,S,\{\mathfrak{p}\}}(0) \\
&= \sum_{\mathfrak{p}\in\Omega} n_{\mathfrak{p}} \cdot n \cdot R_{L/k,w}(\eta_{S,\mathfrak{p}}) \\
&= n \cdot R_{L/k,w}\big(\prod_{\mathfrak{p}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}}}\big) \\
&= n \cdot R_{L/k,w}(\eta_S).
\end{aligned}$$

Noting that η_S is a S_K-unit having the correct absolute values, we just have to show that it satisfies the abelian condition. For that purpose, we use Theorem 2.5. We claim first that $\eta_S^{\gamma - n_\gamma} = \alpha_\gamma^{w_L}$ for all $\gamma \in \Gamma$. On one hand, we have

$$\eta_S^{\gamma - n_\gamma} = \prod_{\mathfrak{p}\in\Omega}\prod_{\mathfrak{q}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p}} n_{\mathfrak{q},\gamma}(1 - \sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q}))},$$

and on the other hand, we have

$$\alpha_\gamma^{w_L} = \prod_{\mathfrak{p}\in\Omega}\prod_{\mathfrak{q}\in\Omega} \eta_{S,\mathfrak{p}}^{n_{\mathfrak{p},\gamma} n_{\mathfrak{q}}(1 - \sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q}))}.$$

If we show that $\eta_{S,\mathfrak{p}}^{1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q})} = \eta_{S,\mathfrak{q}}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})}$ for all $\mathfrak{p}, \mathfrak{q} \in \Omega$, then it would prove our claim. Since

$$(1 - \sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p}))\theta'_{L/k,S,\{\mathfrak{q}\}}(0) = (1 - \sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q}))\theta'_{L/k,S,\{\mathfrak{p}\}}(0),$$

and $R_{L/k,w}$ is an isomorphism of $\mathbb{C}[\Gamma]$-modules when restricted to the $()_{1,S}$-component, we conclude that $\eta_{S,\mathfrak{p}}^{1-\sigma_{\mathfrak{q}}^{-1}\mathbb{N}(\mathfrak{q})}$ and $\eta_{S,\mathfrak{q}}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})}$ differ by a root of unity which is congruent to 1 modulo both $\mathfrak{p}$ and $\mathfrak{q}$. Hence, this root or unity is necessarily 1 and we conclude the desired equality.

The second condition in Theorem 2.5 is a simple computation and left to the reader; thus η_S is w_L-abelian over k.

Q.E.D.

4.4. **Functoriality questions.** In order to prove the usual functoriality properties of our question, we will use the (S,T)-version rather than the S-version since it is simpler to deal with the roots of unity in this context. If desired, the reader will have no problem giving a direct proof of these functoriality properties in the setting of the S-version.

Proposition 4.21. *Let K/k be an abelian extension of number fields and let K' be a subextension: $k \subseteq K' \subseteq K$. Let S and T be finite sets of places satisfying Hypotheses 4.1 and 4.2. The following statements are true.*

1. *S is also a 1-cover for K'/k,*
2. *One has $S_{K',min} \subseteq S_{K,min}$, where the notation should be self-explanatory,*
3. *If $v \in S_{K',min}$ then the truth of $St(K/k,S,T,v)$ implies the veracity of $St(K'/k,S,T,v)$.*

Proof:

We shall decorate all the objects of K'/k with an apostrophe as G', G'_v, etc. Let $\chi \in \widehat{G'}$ and consider $\widetilde{\chi} = \chi \circ \pi \in \widehat{G}$ where $\pi : G \longrightarrow G'$ is the usual projection map. Since S is a 1-cover for K/k, there exists $v \in S$ such that $\widetilde{\chi}(G_v) = 1$. Since $\pi(G_v) = G'_v$, we get $\chi(G'_v) = \widetilde{\chi}(G_v) = 1$, that is $G'_v \subseteq \mathrm{Ker}(\chi)$. This shows that S is also a 1-cover for K'/k.

Next, if $S_{K',min} \neq \varnothing$, let $v \in S_{K',min}$. This means that there exists $\chi \in \widehat{G'}$ such that G'_v is the unique decomposition group included in $\mathrm{Ker}(\chi)$. Remark then that $\widetilde{\chi}(G_v) = \chi(G'_v) = 1$. We claim that $v \in S_{K,min}$ as well. If not then there would exists another place $w \neq v$ such that $\widetilde{\chi}(G_w) = 1$, but then we would get $\chi(G'_w) = 1$, which is a contradiction.

As for the third part, let $L = K^{G_v}$ and $L' = K'^{G'_v}$. We remark that $L \cap K' = L'$ and therefore $n' \,|\, n$ where $n = |G_v| = [K : L]$ and $n' = |G'_v| = [K' : L']$. Now suppose that $St(K/k,S,T,v)$ holds true, that is there exists $\eta \in (E_{L,S,T})_{1,S}$ satisfying $n \cdot R_{L/k,w}(\eta) = \theta'_{L/k,S,T}(0)$. Applying the restriction map $\mathrm{res}_{L/L'}$, we get $n \cdot R_{L'/k,w'}(N_{L/L'}(\eta)) = \theta'_{L'/k,S,T}(0)$. Since $n' \,|\, n$, $n = n's$ for some integer s and therefore $n' \cdot R_{L'/k,w'}(N_{L/L'}(\eta)^s) = \theta'_{L'/k,S,T}(0)$. In other words, $St(K'/k,S,T,v)$ is true with Stark n'-unit $N_{L/L'}(\eta)^s$.

Q.E.D.

Proposition 4.22. *Let K/k be an abelian extension of number fields and S and T be finite sets of places of k satisfying Hypothesis 4.1 and 4.2. Let S' be any other finite set of primes of k containing S such that $S' \cap T = \varnothing$, then the following statements are true.*

1. *S' is also a 1-cover for K/k,*
2. *One has $S'_{min} \subseteq S_{min}$,*
3. *If $v \in S'_{min}$ then the truth of $St(K/k,S,T,v,w)$ implies the veracity of $St(K/k,S',T,v,w)$. More precisely, the Stark n-units are related through the following formula*

$$\eta_{S',T} = \eta_{S,T}^{\alpha}, \text{ where } \alpha = \prod_{\mathfrak{p} \in S' \smallsetminus S} (1 - \sigma_{\mathfrak{p}}^{-1}) \in \mathbb{Z}[\Gamma].$$

Proof:

The first part of the proposition is clear.

If $v \in S'_{min}$, then there exists a character $\chi \in \widehat{G}$ such that v is the unique place in S' for which $G_v \subseteq \mathrm{Ker}(\chi)$. Since S itself is a 1-cover for K/k, we see that $v \in S$ necessarily and thus $v \in S_{min}$. This takes care of the second part.

As for the third part, we have

$$\begin{aligned}\theta'_{L/k,S',T}(0) &= \prod_{\mathfrak{p}\in S'\smallsetminus S}(1-\sigma_{\mathfrak{p}}^{-1})\cdot\theta'_{L/k,S,T}(0)\\ &= \prod_{\mathfrak{p}\in S'\smallsetminus S}(1-\sigma_{\mathfrak{p}}^{-1})\cdot nR_{L/k,w}(\eta_{S,T})\\ &= n\cdot R_{L/k,w}(\eta_{S,T}^{\alpha}),\end{aligned}$$

and thus $\eta_{S',T}=\eta_{S,T}^{\alpha}$.

Q.E.D.

4.5. **Some results.** In [5] Erickson proves Conjecture 3.6 in some special cases. His usual strategy is the following. He writes the usual Stark unit as ε^{α} for some $\alpha\in\mathbb{Z}[\Gamma]$ having the same shape as in Proposition 4.22 and where ε is a Stark unit for a smaller set of primes than S. If S is big enough than Erickson's strategy is to show that these elements α are divisible by n.

In fact, he proved that Question 4.2 has an affirmative answer. We will not repeat his proofs over here, but only refer to his work.

Theorem 4.23. *Let K/k be a finite abelian extension of number fields and S a finite set of places satisfying Hypothesis 4.1. If $v\in S_{min}$, suppose that G_v is cyclic generated by an element $\sigma(v)$. Suppose also that there exists a subset $S'\subseteq S$ containing only unramified finite primes and v and such that S' is a 1-cover for K/k. Let $S_0=S\smallsetminus S'$ and set $S_v=S_0\cup\{v\}$. If the usual abelian rank one Stark conjecture $St(L/k,S_v,v)$ is true where $L=K^{G_v}$, then $St(K/k,S,v)$ has an affirmative answer.*

Proof:

See Theorem 6.1, and Lemmata 6.2, 6.3 in [5].

Q.E.D.

Theorem 4.24. *Let K/k be a finite abelian extension of number fields and let S be a finite set of places satisfying Hypothesis 4.1. Suppose that there exists a subset $S'\subseteq S$ which consists of only unramified finite primes and one infinite real place such that S' is a 1-cover for K/k.*

1. *If v is the unique infinite place in S_{min}, then $St(K^{G_v}/k,\{v\}\cup(S\smallsetminus S'),v)$ implies that $St(K/k,S,v)$ has an affirmative answer.*
2. *If $\mathfrak{p}\in S_{min}$ is a finite unramified prime, then $St(K^{G_{\mathfrak{p}}}/k,\{v,\mathfrak{p}\}\cup(S\smallsetminus S'),\mathfrak{p})$ implies that $St(K/k,S,\mathfrak{p})$ has an affirmative answer.*

Proof:

See Theorem 6.5 in [5].

Q.E.D.

Theorem 4.25. *Let K/k be an abelian extension of number fields and let S be a finite set of places satisfying Hypothesis 4.1 and suppose also that $|S|\geq 3$. Let $\mathfrak{p}$ be a finite prime in S_{min} and set $R=S\smallsetminus\{\mathfrak{p}\}$. For every $\gamma\in\Gamma$, let n_γ be an integer satisfying $\zeta^{\gamma}=\zeta^{n_\gamma}$ for all $\zeta\in\mu_L$. Suppose that*

1. $\frac{w_L\theta_{L/k,R}(0)}{n}\in\mathbb{Z}[\Gamma]$,
2. $\frac{(\gamma-n_\gamma)\theta_{L/k,R}(0)}{n}\in\mathbb{Z}[\Gamma]$ *for all* $\gamma\in\Gamma$.

If $\mathfrak{P}$ is a principal prime ideal of L lying above $\mathfrak{p}$, then $St(K/k,S,\mathfrak{p},\mathfrak{P})$ has an affirmative answer.

Proof:

We use Proposition 4.4. Since $\mathfrak{P}$ is principal, there exists $u\in L^{\times}$ such that $\mathfrak{P}=(u)$. Therefore $\mathfrak{P}^{\frac{w_L\theta_{L/k,R}(0)}{n}}=(u^{\frac{w_L\theta_{L/k,R}(0)}{n}})$. Now $u^{\frac{w_L\theta_{L/k,R}(0)}{n}}$ is an anti-unit and also w_L-abelian over k by Lemmata 4.10 and 4.12.

Q.E.D.

The following theorem takes care of the simplest possible case, namely the biquadratic extensions.

Theorem 4.26. *Let K/k be a biquadratic extension and let S be a 1-cover satisfying Hypothesis 4.1. Then $St(K/k,S,v)$ has an affirmative answer for any $v \in S_{min}$.*

Proof:

If S contains a split prime, we are in the usual setting of the classical abelian rank one Stark conjecture and the result is known due to Sands (see [10] and [11]). Suppose that S does not contain a split prime. Using the fact that $S \neq S_{min}$, we claim that $|S| \geq 4$. Indeed, by Theorem 3.23, there are three minimal cocyclic subgroups for K/k, namely the three subgroups of order two. Since S is a 1-cover, we necessarily have $|S| \geq 3$ by Theorem 3.15. Since $S \neq S_{min}$, we also get by Theorem 3.16 that $|S| \geq 4$ as we claimed. If L/k is a quadratic extension and ε is a Stark unit for the usual abelian rank one Stark conjecture $St(L/k,S,v)$ where v splits completely in L/k, Tate showed in [14], provided that $|S| \geq 3$, the existence of an $\eta \in E_{K,S}$ such that

1. $|\eta|_w = 1$ for all $w \in S_L$ not lying above v,
2. The extension $L(\eta^{1/w_L})/k$ is abelian,
3. A Stark unit ε for $St(L/k,S,v)$ can be taken to be $\varepsilon = \eta^{2^{|S|-3}}$.

Using this last theorem of Tate, we see that $St(K/k,S,v)$ has an affirmative answer for any $v \in S_{min}$.

Q.E.D.

4.5.1. *The place $v \in S_{min}$ is real infinite.* The next theorem is motivated by [3].

Theorem 4.27. *Let K/k be an abelian extension of number fields with Galois group G and let S be a finite set of places satisfying Hypothesis 4.1. Let $v \in S_{min}$ be an infinite real place such that v does not split all the way up to K (otherwise, we would be in the case of the usual abelian rank one Stark conjecture). Let $L = K^{G_v}$ and fix a place $w \in S_L$ lying above v. Assuming Gross's conjecture (including the abelian rank one Stark conjecture), there exist $\eta \in L^\times$ such that*

$$\theta'_{L/k,S}(0) = \frac{2}{w_L} R_{L/k,w}(\eta).$$

Moreover, $|\eta|_{w'} = 1$ for all places w' of L not lying above v.

Proof:

Since w is a real infinite place, the local reciprocity map $\mathrm{rec}_w : L_w^\times \longrightarrow \mathrm{Gal}(K/L)$ is just the map $x \mapsto \mathrm{sgn}_w(x)$ which is 1 if and only if x is positive at w. Let T be any set satisfying Hypothesis 4.2, since S is a 1-cover for K/k, Gross's conjecture implies that $\mathrm{rec}_w(\varepsilon_{S,T}^\gamma) = 1$ for all $\gamma \in \Gamma$. In other words, $\varepsilon_{S,T}^\gamma$ is positive at w for all $\gamma \in \Gamma$.

Let $\varepsilon \in L^\times$ be a Stark unit for $St(L/k,S,v)$. We can assume that ε is positive at w multiplying by -1 if necessary. Take any prime $\mathfrak{p}$ of k which split completely in L/k and is relatively prime to S and w_K. Then $T = \{\mathfrak{p}\}$ satisfies Hypothesis 4.2. Let also $\mathfrak{P}$ be any prime of L lying above $\mathfrak{p}$ and let $\sigma_{\mathfrak{p}} = \left(\frac{L(\sqrt{\varepsilon})/k}{\mathfrak{p}}\right)$ be the Frobenius automorphism. By Theorem 4.19, we have

$$\varepsilon_{S,T} = \sqrt{\varepsilon}^{1-\sigma_{\mathfrak{p}}^{-1}\mathbb{N}(\mathfrak{p})} = \left(\sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-\mathbb{N}(\mathfrak{p})}\right)^{\sigma_{\mathfrak{p}}^{-1}}.$$

Now

$$\sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-\mathbb{N}(\mathfrak{p})} = \sqrt{\varepsilon}^{\sigma_{\mathfrak{p}}-1+1-\mathbb{N}(\mathfrak{p})} = \left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} \varepsilon^{\frac{1-\mathbb{N}(\mathfrak{p})}{2}}.$$

This shows that

$$\varepsilon_{S,T} = \left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} \varepsilon^{\frac{1-\mathbb{N}(\mathfrak{p})}{2}}.$$

Since both $\varepsilon_{S,T}$ and ε are positive at w, we get $\left(\frac{\varepsilon}{\mathfrak{P}}\right)_{L,2} = 1$. By Hasse's principle for powers (Theorem 2.7), we conclude that $\varepsilon = \eta^2$ for some $\eta \in L^\times$. Moreover, $|\eta|_{w'} = 1$ for all places w' of L not lying above v. Finally, we have

$$\theta'_{L/k,S}(0) = \frac{1}{w_L} R_{L/k,w}(\varepsilon) = \frac{2}{w_L} R_{L/k,w}(\eta).$$

Q.E.D.

Because of this last theorem, only one thing is left to investigate (assuming the abelian rank one Stark and the Gross conjectures) in the case where $v \in S_{min}$ is a real infinite place: the abelian condition. So far, we have not found any general strategy to attack this problem. All numerical computations presented in §5 satisfy the abelian condition.

We also remark that in the proof of the last theorem, we used Gross's conjecture in the case where the prime which splits in L/k is an infinite real place. This case of the Gross conjecture when the base field is $\mathbb{Q}$ is known to be true because of classical results (see §1.1 of [3] combined with the usual functorial properties of the Gross conjecture as top change and enlarging the set S). If $k \neq \mathbb{Q}$, and K is a CM-field, then Theorem 4.27 has no content. Otherwise, when the top field is not a CM-field, some progress towards Gross's conjecture has been made by Reid in [8].

4.5.2. *The place $\mathfrak{p} \in S_{min}$ is finite unramified: The extension of the Brumer-Stark conjecture.* As a corollary of Theorem 4.23, we get the following one.

Theorem 4.28. *Let K/k be a finite abelian extension of number fields and S a finite set of places satisfying Hypothesis 4.1. Suppose that there exists a subset $S' \subseteq S$ which consists of only unramified finite primes and such that S' is a 1-cover for K/k. Let $S_0 = S \smallsetminus S'$ and for all $\mathfrak{p} \in S_{min}$, set $S_{\mathfrak{p}} = S_0 \cup \{\mathfrak{p}\}$. If the usual abelian rank one Stark conjecture $St(K^{G_{\mathfrak{p}}}/k, S_{\mathfrak{p}}, \mathfrak{p})$ is true for all $\mathfrak{p} \in S_{min}$, then $St(K/k, S, \mathfrak{p})$ has an affirmative answer for all $\mathfrak{p} \in S_{min}$.*

Proof:

Immediate from Theorem 4.23.

Q.E.D.

We remark that in cases where this theorem applies, S_{min} consists of only finite unramified primes.

Theorem 4.29. *Let K/k be an abelian extension of number fields and let R be a $\mathfrak{p}$-1-cover. For every $\gamma \in \Gamma$, let n_γ be integers satisfying $\zeta^\gamma = \zeta^{n_\gamma}$ for all $\zeta \in \mu_L$. Suppose that*

(1) $\frac{w_L \theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$,

(2) $\frac{(\gamma - n_\gamma)\theta_{L/k,R}(0)}{n} \in \mathbb{Z}[\Gamma]$ *for all $\gamma \in \Gamma$.*

If $h_L = 1$, then $BrSt(K/k, R)$ has an affirmative answer.

Proof:

Immediate from Theorem 4.25.

Q.E.D.

Theorem 4.30. *Let K/k be an abelian extension of number fields with Galois group G and let R be a $\mathfrak{p}$-1-cover. Let T be a finite set of primes of k satisfying Hypothesis 4.2 for the set $S = R \cup \{\mathfrak{p}\}$. As above, set $L = K^{G_{\mathfrak{p}}}$. Then the Gross conjecture implies that*

$$\frac{\theta_{L/k,R,T}(0)}{n} \in \mathbb{Z}[\Gamma],$$

where we recall $\Gamma = \mathrm{Gal}(L/k)$ and $n = |G_{\mathfrak{p}}|$.

Proof:

Let $\mathfrak{P}$ be a prime ideal of L lying above $\mathfrak{p}$. Since $\mathfrak{P}$ is unramified in K/L, the local reciprocity map is simple to described: One has $rec_{\mathfrak{P}}(x) = \sigma_{\mathfrak{p}}^n$, where $n = \mathrm{ord}_{\mathfrak{P}}(x)$ and $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism at $\mathfrak{p}$. We note that

$$\sigma_{\mathfrak{p}} = \left(\frac{K/k}{\mathfrak{p}}\right) = \left(\frac{K/L}{\mathfrak{P}}\right).$$

Suppose that $St(L/k,S,T,\mathfrak{p},\mathfrak{P})$ is true and let $\varepsilon_{S,T}$ be the corresponding Stark unit. Gross's conjecture implies that

$$rec_{\mathfrak{P}}(\varepsilon_{S,T}^{\gamma^{-1}}) = \prod_{\substack{\sigma\in G \\ \sigma|_L=\gamma}} \sigma^{\zeta_{K/k,S,T}(0,\sigma^{-1})},$$

for all $\gamma\in\Gamma$. Since S is a 1-cover for K/k, we have $\theta_{K/k,S,T}(0)=0$; therefore, $\zeta_{K/k,S,T}(0,\sigma)=0$, for all $\sigma\in G$. Hence Gross's conjecture predicts that

$$rec_{\mathfrak{P}}(\varepsilon_{S,T}^{\gamma}) = 1, \tag{10}$$

for all $\gamma\in\Gamma$. On the other hand, we have $\mathrm{ord}_{\mathfrak{P}}(\varepsilon_{S,T}^{\gamma}) = \zeta_{L/k,R,T}(0,\gamma)$, for all $\gamma\in\Gamma$. Thus

$$rec_{\mathfrak{P}}(\varepsilon_{S,T}^{\gamma}) = \sigma_{\mathfrak{p}}^{\zeta_{L/k,R,T}(0,\gamma)}, \tag{11}$$

for all $\gamma\in\Gamma$. Combining equations (10) and (11), we get $f_{\mathfrak{p}} \,|\, \zeta_{L/k,R,T}(0,\gamma)$, for all $\gamma\in\Gamma$, where $f_{\mathfrak{p}}$ is the order of $\sigma_{\mathfrak{p}}$ in G. Since $\mathfrak{p}$ is unramified we have $f_{\mathfrak{p}}=[K:L]=|G_{\mathfrak{p}}|$. At last, since $\theta_{L/k,R,T}(0)=\sum_{\gamma\in\Gamma}\zeta_{L/k,R,T}(0,\gamma)\cdot\gamma^{-1}$, we get the desired result.

Q.E.D.

Remark 4. *Assuming Gross's conjecture, this takes care of point* (1) *of* $BrSt(K/k,R,T)$ *(Question 4.18). Also, combined with Lemma 2.3, it takes care of point* (1) *of* $BrSt(K/k,R)$ *(Question 4.7).*

5. Some numerical computations

The numerical computations presented here are taken from [16] where the reader can find more examples.

Because of Theorem 4.28 and 4.24, we have to find examples for which there exist finite distinguished minimal cocyclic subgroups. For each of the examples below, we give a set S which is a 1-cover and the corresponding S_{min} which have been found with the help of the softwares SAGE [13] and PARI [15]. In order to do so, we computed all minimal cocyclic subgroups of the Galois group and check which ones are distinguished. We completed the set S with appropriate Frobenius automorphisms corresponding to some finite unramified primes. Table 1 shows all $m\not\equiv 2 \pmod 4$ such that $m\le 200$ and for which there exists a finite distinguished minimal cocyclic subgroup for $G_m=\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$.

m	Factorization	Structure of G_m	Distinguished place v
20	$2^2\cdot 5$	$(4,2)$	5
24	$2^3\cdot 3$	$(2,2,2)$	3
40	$2^3\cdot 5$	$(4,2,2)$	5
48	$2^4\cdot 3$	$(4,2,2)$	3
60	$2^2\cdot 3\cdot 5$	$(4,2,2)$	$2,3,5$
68	$2^2\cdot 17$	$(16,2)$	17
80	$2^4\cdot 5$	$(4,4,2)$	5
96	$2^5\cdot 3$	$(8,2,2)$	3
120	$2^3\cdot 3\cdot 5$	$(4,2,2,2)$	$2,3,5$
136	$2^3\cdot 17$	$(16,2,2)$	17
160	$2^5\cdot 5$	$(8,4,2)$	5
171	$3^2\cdot 19$	$(9,3,2,2)$	19
192	$2^6\cdot 3$	$(16,2,2)$	3
195	$3\cdot 5\cdot 13$	$(3,4,4,2)$	5

Table 1. Cyclotomic fields with a distinguished subgroup

If K is an abelian extension of $\mathbb{Q}$, we denote its Galois group by G. Given a prime p or ∞, we denote K^{G_p} more simply by K_p (not to be confused with a completion), and we denote $S\smallsetminus\{p\}$ by R_p. Hence,

$R_p \cup \{p\} = S$, and we also set $n_p = |G_p|$. Moreover, we let $\Gamma_p = G/G_p = \mathrm{Gal}(K_p/\mathbb{Q})$. For each of the field K_p where $p \in S_{min}$, we give the class number, the number of roots of unity of K_p, the cardinality of G_p, the degree of K_p over $\mathbb{Q}$, the R_p-equivariant L-function of $K_p/\mathbb{Q}$ evaluated at 0 which is an element of $\mathbb{Q}[\Gamma_p]$, and the greatest common divisor of the coefficients of $\theta_{K_p/\mathbb{Q},R_p}(0)$ if it already lies in $\mathbb{Z}[\Gamma_p]$. Given any $\alpha \in \mathbb{Z}[G]$, we will denote the gcd of the coefficients of α by $\gamma(\alpha)$. For each of the field K_p when p is a finite ramified prime, we choose a sample of primes t such that $\{t\} \cap S = \varnothing$ and $\mu_{K,\{t\}} = 1$, and we check point (1) of Proposition 4.16 for the element $\theta_{K_p/\mathbb{Q},R_p,\{t\}}(0)$. If $\infty \in S_{min}$, we check that the Stark unit ε for $St(K_\infty/\mathbb{Q}, S, \infty)$ is a square in K_∞ and if $\varepsilon_\infty = \eta_\infty^2$ then we also check that η_∞ is 2-abelian over $\mathbb{Q}$, i.e. that $K_\infty(\sqrt{\eta_\infty})/\mathbb{Q}$ is an abelian extension of number fields.

5.1. **The field** $K = \mathbb{Q}(\zeta_{40})$**.** Let $S = \{\infty, 2, 5, 11, 19, 23\}$. In this case, $S_{min} = \{\infty, 5, 11, 19, 23\}$ and we computed the following data:

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_5	1	4	8	2	$[2, -2]$	2
K_{11}	1	10	2	8	$[3, 1, -1, -3, 1, -1, 3, -3]/5$	-
K_{19}	2	2	2	8	$[-1, -1, -1, 1, 1, 1, 1, -1]$	1
K_{23}	2	2	4	4	$[2, 2, -2, -2]$	2

(1) In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus $\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. From the table below,

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
3	8
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$
17	$32 = 8 \cdot 4$
3389	$6776 = 8 \cdot 847$

we see that we always have $\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. Moreover, since $h_{K_5} = 1$, we conclude that $St(K/\mathbb{Q}, S, 5)$ is true by Theorem 4.25.

(2) In K_∞, we found numerically that the minimal polynomial of ε_∞ is $q(x) = x^2 - 1442x + 1$. We also have $\varepsilon_\infty = \eta_\infty^2$, as predicted by Theorem 4.27, where
$$\eta_\infty = \frac{1}{38}\varepsilon_\infty + \frac{1}{38}.$$
Furthermore, η_∞ itself is a square in $\mathbb{Q}(\varepsilon_\infty)$:
$$\eta_\infty = \left(\frac{1}{228}\varepsilon_\infty - \frac{37}{228}\right)^2.$$
Hence, the abelian condition is satisfied.

(3) In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus $\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}]$, but we already know this by Theorem 4.30. Since $h_{K_{11}} = 1$, we also conclude that $BrSt(K/\mathbb{Q}, R_{11})$ is true by Theorem 4.29.

(4) In K_{19} one has $\gamma(w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)) = 2$, thus $\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}} \in \mathbb{Z}[\Gamma_{19}]$, but we already know this by Theorem 4.30. We chose a prime $\mathfrak{p}$ of K_{19} lying above 19. It turns out that it is not principal. We computed $\mathfrak{p}^{\frac{w_{K_{19}} \cdot \theta_{K_{19}/\mathbb{Q},R_{19}}(0)}{n_{19}}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_{19}(\alpha^{1/2})/\mathbb{Q}$

is abelian: Its Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Since $h_{K_{19}} = 2$, we conclude that $BrSt(K/\mathbb{Q}, R_{19})$ is true.

(5) In K_{23} one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 4$, thus $\frac{w_{K_{23}}\cdot\theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}]$, but we already know this by Theorem 4.30. We chose a prime $\mathfrak{p}$ of K_{23} lying above 23. It turns out that it is not principal. We computed $\mathfrak{p}^{\frac{w_{K_{23}}\cdot\theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_{23}(\alpha^{1/2})/\mathbb{Q}$ is abelian: Its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Since $h_{K_{23}} = 2$, we conclude that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

5.2. **The field $K = \mathbb{Q}(\zeta_{60})$.** Let $S = \{\infty, 2, 3, 5, 11, 43\}$. In this case, $S_{min} = \{\infty, 2, 5, 11, 43\}$ and we computed the following data:

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_2	2	2	8	2	$[4, -4]$	4
K_5	1	4	8	2	$[2, -2]$	2
K_{11}	2	10	2	8	$[1, -7, -3, -1, 1, 3, 7, -1]/5$	–
K_{43}	2	6	4	4	$[2, -2, 2, -2]/3$	–

(1) In K_2 one has $\gamma(w_{K_2} \cdot \theta_{K_2/\mathbb{Q},R_2}(0)) = 8$, thus $\frac{w_{K_2}\cdot\theta_{K_2/\mathbb{Q},R_2}(0)}{n_2} \in \mathbb{Z}[\Gamma_2]$. From the table below,

t	$\gamma(\theta_{K_2/\mathbb{Q},R_2,\{t\}}(0))$
7	$32 = 8 \cdot 4$
13	$56 = 8 \cdot 7$
17	$64 = 8 \cdot 8$
31	$120 = 8 \cdot 15$
3389	$13560 = 8 \cdot 1695$

we see that we always have $\frac{\theta_{K_2/\mathbb{Q},R_2,\{t\}}(0)}{n_2} \in \mathbb{Z}[\Gamma_2]$. We chose a prime $\mathfrak{p}$ of K_2 lying above 2. It turns out that it is not principal. We computed $\mathfrak{p}^{\frac{w_{K_2}\cdot\theta_{K_2/\mathbb{Q},R_2}(0)}{n_2}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_2(\alpha^{1/2})/\mathbb{Q}$ is abelian: Its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We conclude that $St(\mathbb{Q}(\zeta_{60})/\mathbb{Q}, S, 2)$ is true.

(2) In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 8$, thus $\frac{w_{K_5}\cdot\theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. From the table below,

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
7	$16 = 8 \cdot 2$
13	$24 = 8 \cdot 3$
17	$32 = 8 \cdot 4$
31	$64 = 8 \cdot 8$
3389	$6776 = 8 \cdot 847$

we see that we always have $\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. Moreover, since $h_{K_5} = 1$, we conclude that $St(\mathbb{Q}(\zeta_{60})/\mathbb{Q}, S, 5)$ is true by Theorem 4.25.

(3) In K_∞, we found numerically that the minimal polynomial of ε_∞ is

$$x^8 - 538\,x^7 + 14763\,x^6 - 71876\,x^5 + 118325\,x^4 - 71876\,x^3 + 14763\,x^2 - 538\,x + 1,$$

hence $\mathbb{Q}(\varepsilon_\infty) = K_\infty$. As predicted by Theorem 4.27, we have $\varepsilon_\infty = \eta_\infty^2$ where

$$\eta_\infty = \frac{24919}{130332851}\varepsilon_\infty^7 - \frac{53631771}{521331404}\varepsilon_\infty^6 + \frac{12188357}{4308524}\varepsilon_\infty^5 - \frac{3627530725}{260665702}\varepsilon_\infty^4 + \frac{6129121873}{260665702}\varepsilon_\infty^3 - \frac{33221103}{2154262}\varepsilon_\infty^2 + \frac{2404381171}{521331404}\varepsilon_\infty + \frac{18410903}{521331404}.$$

In this case, η_∞ itself is not a square in K_∞, but we checked that the abelian condition is satisfied, i.e. $K_\infty(\eta_\infty^{1/2})/\mathbb{Q}$ is an abelian extension of number fields.

(4) In K_{11} one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 2$, thus $\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}]$, but we already know this by Theorem 4.30. We chose a prime $\mathfrak{p}$ of K_{11} lying above 11. It turns out that it is not principal. We computed $\mathfrak{p}^{\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_{11}(\alpha^{1/10})/\mathbb{Q}$ is abelian: Its Galois group is isomorphic to $\mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $h_{K_{11}} = 2$, we conclude that $BrSt(K/\mathbb{Q}, R_{11})$ is true.

(5) In K_{43} one has $\gamma(w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)) = 4$, thus $\frac{w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}]$, but we already know this by Theorem 4.30. We chose a prime $\mathfrak{p}$ of K_{43} lying above 43. It turns out that it is not principal. We computed $\mathfrak{p}^{\frac{w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)}{n_{43}}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_{43}(\alpha^{1/6})/\mathbb{Q}$ is abelian: Its Galois group is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $h_{K_{43}} = 2$, we conclude that $BrSt(K/\mathbb{Q}, R_{43})$ is true.

5.3. **Example 3.** Here is a particular example which is given in [4]. Let p, q be two odd prime numbers satisfying $p \equiv 1 \pmod 4$, $\quad q \equiv 3 \pmod 4$, and $\left(\frac{p}{q}\right) = 1$. Let $K' = \mathbb{Q}(\zeta_q)^{D_p}$ where D_p is the decomposition group associated to p in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. Remark that $\mathbb{Q}(\sqrt{-q}) \subseteq K'$. Let K'^+ be the maximal real subfield of K'. Let l be an odd prime number, different from p and q and satisfying

$$\left(\frac{K'/\mathbb{Q}}{l}\right) \neq 1, \ \left(\frac{K'^+/\mathbb{Q}}{l}\right) = 1, \text{ and } \left(\frac{p}{l}\right) = -1,$$

(i.e. l splits completely in $K'^+/\mathbb{Q}$, but does not split completely in $K'/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$). Let $K = K'(\sqrt{p})$. Then $S = \{\infty, p, q, l\}$ is a 1-cover for $\widehat{G}$ and $S_{min} = \{\infty, p, l\}$. Indeed, we have

$$\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \times \mathrm{Gal}(K'/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z},$$

where $m = (q-1)/2f_p$, and f_p is the inertia index of p is $\mathbb{Q}(\zeta_q)$. Remark that since $q \equiv 3 \pmod 4$ we have $(m, 2) = 1$. Therefore, we have by Theorem 3.23 and Corollary 3.22 that the minimal cocyclic subgroups are precisely the 3 subgroups of cardinality 2 of G. We then use Theorem 3.15 and Theorem 3.16 in order to conclude that $S = \{\infty, p, q, l\}$ is a 1-cover for $\widehat{G}$ and $S_{min} = \{\infty, p, l\}$. Let us take the pair $p = 5$, $q = 31$ and $l = 37$. We computed the following data:

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_5	3	2	2	10	$[1, -1, -1, 1, -1, 1, -1, 1, 1, -1]$	1
K_{37}	64	2	2	10	$[0, 0, 0, 2, 0, -2, 0, 0, 0, 0]$	2

(1) In K_5 one has $\gamma(w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)) = 2$, thus $\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. From the table below, we see that we always have $\frac{\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0)}{n_5} \in \mathbb{Z}[\Gamma_5]$. The class group turns out to be cyclic of order 3. We chose a prime $\mathfrak{p}$ of K_5 whose class generates the whole class group. We computed $\mathfrak{p}^{\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_5(\alpha^{1/2})/\mathbb{Q}$ is abelian: Its Galois group is isomorphic to $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We conclude that the whole class group is annihilated by $\frac{w_{K_5} \cdot \theta_{K_5/\mathbb{Q},R_5}(0)}{n_5}$, so in particular $St(K/\mathbb{Q}, S, 5)$ holds true.

t	$\gamma(\theta_{K_5/\mathbb{Q},R_5,\{t\}}(0))$
3	2
7	2
11	2
13	2
17	2

(2) In K_∞, we computed the minimal polynomial of ε_∞, and we obtained

$$\begin{aligned} p(X) =& x^{10} - 242015x^9 + 449115185x^8 - 108113455735x^7 + 4075762267865x^6 - \\ & 40560353573727x^5 + 4075762267865x^4 - 108113455735x^3 + \\ & 449115185x^2 - 242015x + 1. \end{aligned}$$

We checked that it is a square in K_∞ as predicted by Theorem 4.27. Moreover, we checked that $K_\infty(\eta_\infty^{1/2})/\mathbb{Q}$ is abelian.

(3) In K_{37}, one has $\gamma(w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)) = 4$, thus we have $\frac{w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)}{n_{37}} \in \mathbb{Z}[\Gamma_{37}]$, but we already know this by Theorem 4.30. The class group turns out to be isomorphic to

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We took generators $\mathfrak{p}_i$ for the class group of K_{37}. We computed $\mathfrak{p}_i^{\frac{w_{K_{37}} \cdot \theta_{K_{37}/\mathbb{Q},R_{37}}(0)}{n_{37}}}$, and each time we obtained a principal ideal whose generator, say α_i, was already an anti-unit. Then we checked that the extension $K_{37}(\alpha_i^{1/2})/\mathbb{Q}$ is abelian for all i. We conclude that $BrSt(K/\mathbb{Q}, R_{37})$ is true.

5.4. **Example 4.** Take the primes 7 and 43. Since $43 \equiv 1 \pmod 3$, let L be the unique subfield of $\mathbb{Q}(\zeta_{43})$ which is of degree 3 over $\mathbb{Q}$ and set $K = \mathbb{Q}(\zeta_7) \cdot L$. It is an abelian extension of $\mathbb{Q}$ with Galois group $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. There are four minimal cocyclic subgroups by Theorem 3.23 and they are precisely the four subgroups of order three. The set $S = \{\infty, 7, 43, 11, 23, 67\}$ is a 1-cover and $S_{min} = \{43, 11, 23, 67\}$. There are only two ramified primes in K, namely 7 and 43. We computed the following data

Fields	h_{K_p}	w_{K_p}	$n_p = [K : K_p]$	$[K_p : \mathbb{Q}]$	$\theta_{K_p/\mathbb{Q},R_p}(0)$	gcd
K_{43}	1	14	3	6	$[3, -9, -6, 6, 9, -3]/7$	–
K_{11}	57	2	3	6	$[-6, -15, -9, 9, 15, 6]$	3
K_{23}	9	2	3	6	$[-3, 0, 0, 3, 3, -3]$	3
K_{67}	9	2	3	6	$[3, -3, -3, 0, 0, 3]$	3

(1) In K_{43} one has $\gamma(w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)) = 6 = 2 \cdot 3$, thus $\frac{w_{K_{43}} \cdot \theta_{K_{43}/\mathbb{Q},R_{43}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}]$. From the table below, we see that we always have $\frac{\theta_{K_{43}/\mathbb{Q},R_{43},\{t\}}(0)}{n_{43}} \in \mathbb{Z}[\Gamma_{43}]$. Moreover, since $h_{K_{43}} = 1$, we conclude that $St(K/\mathbb{Q}, S, 43)$ is true by Theorem 4.25.

t	$\gamma(\theta_{K_{43}/\mathbb{Q},R_{43},\{t\}}(0))$
3	3
5	3
13	3
17	3
29	12

(2) In K_{11}, one has $\gamma(w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)) = 6$, thus we have $\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}} \in \mathbb{Z}[\Gamma_{11}]$, but we already know this by Theorem 4.30. The class group turns out to be cyclic of order 57. We chose a prime $\mathfrak{p}$ of K_{11} whose class generates the whole class group. We computed $\mathfrak{p}^{\frac{w_{K_{11}} \cdot \theta_{K_{11}/\mathbb{Q},R_{11}}(0)}{n_{11}}}$, and we obtained a principal ideal whose generator, say α, was already an anti-unit. Then we checked that the extension $K_{11}(\alpha^{1/2})/\mathbb{Q}$ is abelian: In fact $K_{11}(\alpha^{1/2}) = K_{11}$ in this case. We conclude that $BrSt(K/\mathbb{Q}, R_{11})$ is true.

(3) In K_{23}, one has $\gamma(w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)) = 6$, thus we have $\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}} \in \mathbb{Z}[\Gamma_{23}]$, but we already know this by Theorem 4.30. The class group turns out to be isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. We took generators $\mathfrak{p}_i$ for the class group of K_{23} $(i = 1, 2)$. We computed $\mathfrak{p}_i^{\frac{w_{K_{23}} \cdot \theta_{K_{23}/\mathbb{Q},R_{23}}(0)}{n_{23}}}$, and each time we obtained a principal ideal whose generator, say α_i, was already an anti-unit. Then we checked that the extension $K_{23}(\alpha_i^{1/2})/\mathbb{Q}$ is abelian for all i: In fact $K_{23}(\alpha_i^{1/2}) = K_{23}$ for $i = 1, 2$. We conclude that $BrSt(K/\mathbb{Q}, R_{23})$ is true.

(4) In K_{67}, one has $\gamma(w_{K_{67}} \cdot \theta_{K_{67}/\mathbb{Q},R_{67}}(0)) = 6$, thus we have $\frac{w_{K_{67}} \cdot \theta_{K_{67}/\mathbb{Q},R_{67}}(0)}{n_{67}} \in \mathbb{Z}[\Gamma_{67}]$, but we already know this by Theorem 4.30. The class group turns out to be isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. We took generators $\mathfrak{p}_i$ for the class group of K_{67} $(i = 1, 2)$. We computed $\mathfrak{p}_i^{\frac{w_{K_{67}} \cdot \theta_{K_{67}/\mathbb{Q},R_{67}}(0)}{n_{67}}}$, and each time we obtained a principal ideal whose generator, say α_i, was already an anti-unit. In fact $\alpha_1 = 1$. Then we checked that the extension $K_{67}(\alpha_2^{1/2})/\mathbb{Q}$ is abelian: In fact $K_{67}(\alpha_2^{1/2}) = K_{67}$. We conclude that $BrSt(K/\mathbb{Q}, R_{67})$ is true.

6. Conclusion

Question 4.2 seems to be easier to investigate than Conjecture 3.6 both numerically and theoretically. Nevertheless, the reader should keep in mind that Conjecture 3.6 is only implied by an affirmative answer to Question 4.2, but they are not equivalent. If a negative answer to Question 4.2 is found, it would mean that there are some mysterious combinations of various Stark units coming from intermediate fields provided Conjecture 3.6 still holds true.

We refrain from calling Question 4.2 a conjecture, since in the opinion of the author, there is not enough evidence for a positive answer to it. In fact, Stark himself would rather call Conjecture 3.6 a question...The ultimate goal is to find a precise statement for what is happening when there is no split prime in the set S.

There are several interesting questions which we believe deserve attention. The function field case should be studied as well. Linking Conjecture 3.6 or Question 4.2 with the Equivariant Tamagawa Number Conjecture would be really interesting. Would the recent work of Greither and Popescu on an equivariant main conjecture in Iwasawa theory be of any use here as it is for the conjectures of

Brumer-Stark, Rubin-Stark and Coates-Sinnott? Given a finite abelian extension of number fields L/k, a finite set of primes $R \supseteq S(L/k)$ and an integer n, are there conditions which guarantee the existence of a prime $\mathfrak{p}$ of k and of an abelian extension K/k such that $L \subseteq K$, $G_{\mathfrak{p}} = \mathrm{Gal}(K/L)$, $n = |G_{\mathfrak{p}}|$, $S = R \cup \{\mathfrak{p}\}$ is a 1-cover for K/k and $\mathfrak{p} \in S_{min}$? If yes, this would give information on the greatest common divisor of the coefficients of $w_L \cdot \theta_{L/k,S}(0)$. Let us end this list by saying that this conjecture is not yet known for all abelian extensions of $\mathbb{Q}$.

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