ON ABELIAN ℓ-TOWERS OF MULTIGRAPHS II

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Abstract. Let ℓ be a rational prime. Previously, abelian ℓ-towers of multigraphs were introduced which are analogous to $\mathbb{Z}_\ell$-extensions of number fields. It was shown that for a certain class of towers of bouquets, the growth of the ℓ-part of the number of spanning trees behaves in a predictable manner (analogous to a well-known theorem of Iwasawa for $\mathbb{Z}_\ell$-extensions of number fields). In this paper, we give a generalization to a broader class of regular abelian ℓ-towers of bouquets than was originally considered. To carry this out, we observe that certain shifted Chebyshev polynomials are members of a continuously parametrized family of power series with coefficients in $\mathbb{Z}_\ell$ and then study the special value at $u = 1$ of the Artin-Ihara L-function ℓ-adically.

Contents

1. Introduction

In [9], abelian ℓ-towers of multigraphs were introduced which can be viewed as being analogous to $\mathbb{Z}_\ell$-extensions of number fields, where ℓ is a rational prime. To every tuple in $\mathbb{Z}_\ell^t$ (with $t \in \mathbb{N}$ and not all entries divisible by ℓ) one can associate an abelian ℓ-tower of a bouquet with t loops. Furthermore, it was proved that when the tuple belongs to $\mathbb{Z}^t$, the ℓ-adic valuation of the number of spanning trees behaves similarly to the ℓ-adic valuation of the class numbers in $\mathbb{Z}_\ell$-extensions of number fields, as in a well-known theorem of Iwasawa (see Theorem 11 in [1] and also §4.2 of [2]). More specifically, if κ_n denotes the number of spanning trees of the multigraph at the n-th level, then there exist non-negative integers μ, λ, n_0 and an integer ν such that

$$\mathrm{ord}_\ell(\kappa_n) = \mu\ell^n + \lambda n + \nu$$

for $n \geq n_0$. In the present paper, we extend this result to all of $\mathbb{Z}_\ell^t$, thereby generalizing Theorem 5.6 of [9] to a broader class of regular abelian ℓ-towers of bouquets than was originally considered.

The paper is organized as follows. In §2, we observe that the coefficients of the shifted Chebyshev polynomials $P_a(T)$ employed in [9] satisfy some congruences, and in §3 we use this to show that there exists a continuous function

$$f : \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell[\![T]\!],$$

satisfying $f(a) = P_a(T)$ for $a \in \mathbb{N}$. This allows us, in §4, to study ℓ-adically the special value at $u = 1$ of Artin-Ihara L-functions associated to the covers arising in abelian ℓ-towers of a bouquet, which leads to our main result (see Theorem 4.1). In §5, we end the paper with a few examples.

Date: August 5, 2021.
2010 *Mathematics Subject Classification.* Primary: 05C25; Secondary: 11A07, 33C45.

Acknowledgement. We would like to thank Thomas Mattman and John Lind for stimulating discussions and helpful suggestions.

2. Shifted Chebyshev polynomials

Throughout this paper, $\mathbb{N}$ will denote the set of positive integers. Recall that for $a \in \mathbb{Z}_{\geq 0}$ the Chebyshev polynomial (of the first type) $T_a(X) \in \mathbb{Z}[X]$ is the unique polynomial satisfying

$$T_a(\cos(\theta)) = \cos(a\theta)$$

for all $\theta \in \mathbb{R}$.

In §5.4 of [9], some polynomials $P_a(X) \in \mathbb{Z}[X]$ were defined recursively as follows. For $a = 0, 1$, one sets $P_0(X) = 0$, $P_1(X) = X$, and for $a \geq 2$

$$P_a(X) = X(a^2 - (a-1)P_1(X) - (a-2)P_2(X) - \ldots - P_{a-1}(X)).$$

Given $m \in \mathbb{N}$, we let $\zeta_m = \exp(2\pi i/m)$. Furthermore, if $a \in \mathbb{Z}$, we set

$$\varepsilon_m(a) = (1 - \zeta_m^a)(1 - \zeta_m^{-a}) \in \overline{\mathbb{Q}} \subseteq \mathbb{C},$$

and we write ε_m rather than $\varepsilon_m(1)$. The polynomials $P_a(X)$ satisfy various properties, but most notably Lemma 5.5 of [9] shows that for $m \in \mathbb{N}$ and $a \in \mathbb{Z}_{\geq 0}$, one has

$$P_a(\varepsilon_m) = \varepsilon_m(a). \tag{1}$$

Consider now the polynomial $Q_a(X) = 2 - 2 \cdot T_a\left(1 - \frac{X}{2}\right)$. Since

$$\varepsilon_m(a) = 2 - 2\cos\left(\frac{2\pi a}{m}\right),$$

we have $P_a(\varepsilon_m) = Q_a(\varepsilon_m) = \varepsilon_m(a)$ for all $m \in \mathbb{N}$. Since $P_a(X)$ and $Q_a(X)$ agree at infinitely many points, they are equal. Thus, the precise relationship between the polynomials $P_a(X)$ and the Chebyshev polynomials is

$$P_a(X) = 2 - 2 \cdot T_a\left(1 - \frac{X}{2}\right), \tag{2}$$

so that $P_a(X)$ is a shifted Chebyshev polynomial. The polynomials $P_a(X)$ have no constant coefficients and have degree a. From now on, we write

$$P_a(X) = d_1(a)X + d_2(a)X^2 + \ldots + d_a(a)X^a.$$

In order to simplify the notation, we shall also make use of the falling and raising factorials

$$(X)_n = X(X-1)\ldots(X-n+1) \text{ and } X^{(n)} = X(X+1)\ldots(X+n-1).$$

Proposition 2.1. *For $k, n \in \mathbb{N}$, we have*

$$d_k(n) = \begin{cases} (-1)^{k-1}\binom{n+k-1}{2k-1}\frac{n}{k}, & \text{if } k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Our starting point (see §4.21 of [6]) is the equality between Chebyshev polynomials and hypergeometric functions which gives

$$\begin{aligned} T_n(X) &= {}_2F_1\left(-n, n; \frac{1}{2}, \frac{1-X}{2}\right) \\ &= \sum_{i=0}^{\infty} \frac{(-n)^{(i)} n^{(i)}}{\left(\frac{1}{2}\right)^{(i)}} \left(\frac{1}{i!}\right)\left(\frac{1-X}{2}\right)^i \\ &= 1 + n\sum_{i=1}^{n} (-2)^i \frac{(n+i-1)!}{(n-i)!(2i)!}(1-X)^i. \end{aligned}$$

Combining with (2) gives the desired result. ∎

In particular, we have

$$d_1(n) = n^2 \text{ and } d_n(n) = (-1)^{n-1}.$$

The integers $d_k(n)$ satisfy certain congruence relations which we state in Proposition 2.3 below, but we first need a lemma.

Lemma 2.2. *Let ℓ be a rational prime satisfying $\ell \geq 3$. Then for all $t \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{ord}_\ell((2t+1)!(t+1)) \leq t$$

and

$$\mathrm{ord}_2((2t+1)!(t+1)) \leq 2t.$$

Proof. Let ℓ be an arbitrary prime number. The claim is true if $t = 0$ so we can assume that $t \geq 1$. To simplify the notation, let $v = \mathrm{ord}_\ell(t+1)$ and $N = \lfloor \log_\ell(2t+1) \rfloor$. Note first that

$$v \leq \lfloor \log_\ell(t+1) \rfloor \leq N.$$

Using Legendre's formula, we have

$$\mathrm{ord}_\ell((2t+1)!(t+1)) = \sum_{k=1}^{v} \left\lfloor \frac{2t+1}{\ell^k} \right\rfloor + \sum_{k=v+1}^{N} \left\lfloor \frac{2t+1}{\ell^k} \right\rfloor + v,$$

where the first sum is interpreted as being zero when $v = 0$ and the second sum is interpreted as being zero when $v = N$. Now, we have

$$\begin{aligned}
\mathrm{ord}_\ell((2t+1)!(t+1)) &\leq \sum_{k=1}^{v} \left\lfloor \frac{2t+1}{\ell^k} \right\rfloor + \sum_{k=v+1}^{N} \frac{2t+1}{\ell^k} + v \\
&= \sum_{k=1}^{v} \left\lfloor \frac{2(t+1)}{\ell^k} - \frac{1}{\ell^k} \right\rfloor + \sum_{k=v+1}^{N} \frac{2t+1}{\ell^k} + v \\
&= \sum_{k=1}^{v} \frac{2(t+1)}{\ell^k} + \sum_{k=1}^{v} \left\lfloor -\frac{1}{\ell^k} \right\rfloor + \frac{2t+1}{(\ell-1)\cdot \ell^v}\left(1 - \frac{1}{\ell^{N-v}}\right) + v \\
&= \sum_{k=1}^{v} \frac{2(t+1)}{\ell^k} + \frac{2t+1}{(\ell-1)\cdot \ell^v}\left(1 - \frac{1}{\ell^{N-v}}\right) \\
&= \frac{2(t+1)}{\ell-1}\left(1 - \frac{1}{\ell^v}\right) + \frac{2t+1}{(\ell-1)\cdot \ell^v}\left(1 - \frac{1}{\ell^{N-v}}\right).
\end{aligned}$$

If $\ell \geq 3$, then we continue as follows

$$\begin{aligned}
&\leq (t+1)\left(1 - \frac{1}{\ell^v}\right) + \left(t + \frac{1}{2}\right)\frac{1}{\ell^v} \\
&= (t+1) - \frac{1}{2\cdot \ell^v} \\
&< t+1,
\end{aligned}$$

whereas if $\ell = 2$, we rather continue as follows

$$\begin{aligned}
&= 2(t+1)\left(1 - \frac{1}{2^v}\right) + (2t+1)\left(\frac{1}{2^v} - \frac{1}{2^N}\right) \\
&= 2t + 2 - \frac{1}{2^v} - \frac{2t+1}{2^N} \\
&< 2t+1,
\end{aligned}$$

since

$$\frac{2t+1}{2^N} \geq 1.$$

This ends the proof. □

Proposition 2.3. *Let ℓ be any rational prime and let $m, n \in \mathbb{N}$ be such that*

$$m \equiv n \pmod{\ell^s}$$

for some $s \in \mathbb{N}$. Then, we have

$$d_{t+1}(m) \equiv d_{t+1}(n) \pmod{\ell^{s-t}},$$

for $t = 0, 1, \dots, s-1$.

Proof. Without lost of generality, let us assume that $n \geq m$. Assuming first $\ell \geq 3$, it follows from Proposition 2.1 that for $a \geq t+1$, we have

$$d_{t+1}(a) = (-1)^t \frac{(a+t)_{2t+1}}{(2t+1)!} \frac{a}{t+1}. \tag{3}$$

Write $n = m + q\ell^s$ for some integer q. If $m \leq n < t+1$, then the claim is obviously true. If now $m < t+1 \leq n$, then we have to show

$$d_{t+1}(n) \equiv 0 \pmod{\ell^{s-t}}. \tag{4}$$

By (3), we have

$$d_{t+1}(n) = (-1)^t \frac{(m + q\ell^s + t)_{2t+1}}{(2t+1)!} \frac{n}{t+1}.$$

Since $t + 1 > m$, we have $0 \leq m + t \leq 2t$ and thus $(m + q\ell^s + t)_{2t+1} \equiv 0 \pmod{\ell^s}$. Combining this with Lemma 2.2 gives

$$\mathrm{ord}_\ell(d_{t+1}(n)) \geq s - t,$$

which implies (4).

If we have $1 \leq t+1 \leq m \leq n$, then

$$d_{t+1}(n) - d_{t+1}(m) = \frac{(-1)^t}{t+1} \left(m \left(\binom{n+t}{2t+1} - \binom{m+t}{2t+1} \right) + \binom{n+t}{2t+1} q\ell^s \right), \tag{5}$$

but

$$\binom{n+t}{2t+1} - \binom{m+t}{2t+1} = \frac{(m + q\ell^s + t)_{2t+1} - (m+t)_{2t+1}}{(2t+1)!}.$$

Now, $(m + q\ell^s + t)_{2t+1} \equiv (m+t)_{2t+1} \pmod{\ell^s}$. Therefore combining this with (5) and Lemma 2.2 gives the desired congruence as well.

If $\ell = 2$, we first consider $m < t+1 \leq n$. In this case, one of the terms in the product

$$(m + q2^s + t)_{2t+1} = (m + q2^s + t) \dots (m + q2^s - t),$$

is equal to $q2^s$, and moreover, there are at least t even terms and at least 2 terms divisible by 4 when $t \geq 2$. This gives

$$\mathrm{ord}_2((m + q2^s + t)_{2t+1}) \geq s + t$$

unless $t = 2$ in which case $\mathrm{ord}_2((m + q2^s + t)_{2t+1}) \geq s + t - 1$. Combining this with Lemma 2.2 and the equality $\mathrm{ord}_2(5! \cdot 3) = 3$ gives (4).

At last, we consider $1 \leq t+1 \leq m \leq n$ where

$$m \left(\binom{n+t}{2t+1} - \binom{m+t}{2t+1} \right) = \frac{m(m + q2^s + t)_{2t+1} - m(m+t)_{2t+1}}{(2t+1)!}.$$

It suffices to prove that

$$\mathrm{ord}_2(m(m + q2^s + t)_{2t+1} - m(m+t)_{2t+1}) \geq s + t.$$

The product $m(m+t)_{2t+1}$ contains exactly $t+1$ even integers, as does the product $m(m+t+q2^s)_{2t+1}$. After factoring 2^{t+1} out of each product, the remaining difference is visibly congruent to zero mod 2^{s-1}; indeed, a typical term from the first product looks like $(m+j)/2$ and the corresponding term in the second product looks like $(m+j)/2 + q2^{s-1}$. Again combining this with (5) and Lemma 2.2 gives the desired congruence.

□

3. ℓ-ADIC LIMITS OF SHIFTED CHEBYSHEV POLYNOMIALS

Our main references for this section are [2], [4] and [5]. We start by fixing a rational prime ℓ. The unital commutative ring $\mathbb{Z}_\ell[\![T]\!]$ of power series with coefficient in $\mathbb{Z}_\ell$ is a local ring, and its unique maximal ideal is given by $\mathfrak{m} = (\ell, T)$. We view $\mathbb{Z}_\ell[\![T]\!]$ with its usual topology given by the filtration

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \ldots \supseteq \mathfrak{m}^n \supseteq \ldots$$

For $Q \in \mathbb{Z}_\ell[\![T]\!]$, let

$$||Q||_\ell = \begin{cases} \ell^{-v}, & \text{if } Q \neq 0; \\ 0, & \text{if } Q = 0, \end{cases}$$

where $v = \max\{n \in \mathbb{Z}_{\geq 0} \,|\, Q \in \mathfrak{m}^n\}$ with the understanding that $\mathfrak{m}^0$ is the full ring. Then, for $Q, R \in \mathbb{Z}_\ell[\![T]\!]$, the function $|| \cdot ||_\ell : \mathbb{Z}_\ell[\![T]\!] \longrightarrow \mathbb{R}_{\geq 0}$ satisfies:

(1) $||Q||_\ell = 0$ if and only if $Q = 0$,
(2) $||Q \cdot R||_\ell \leq ||Q||_\ell \cdot ||R||_\ell$,
(3) $||Q + R||_\ell \leq ||Q||_\ell + ||R||_\ell$.

Thus, the pair $(\mathbb{Z}_\ell[\![T]\!], d)$, where $d(Q, R) = ||Q - R||_\ell$, is a metric space. As such, $\mathbb{Z}_\ell[\![T]\!]$ is a complete local ring.

We have a function $f : \mathbb{N} \longrightarrow \mathbb{Z}_\ell[\![T]\!]$ defined by

$$n \mapsto f(n) = P_n(T).$$

Proposition 3.1. *The function f is uniformly continuous when $\mathbb{N}$ is endowed with the ℓ-adic topology.*

Proof. We have to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $m, n \in \mathbb{N}$ are such that $|m - n|_\ell < \delta$, then $||P_m(T) - P_n(T)||_\ell < \varepsilon$. It follows from Proposition 2.3 that given any $N \in \mathbb{N}$ and any $m, n \in \mathbb{N}$ satisfying

$$\mathrm{ord}_\ell(m - n) \geq N,$$

one has

$$P_m(T) - P_n(T) \in \mathfrak{m}^{N+1}.$$

In other words, if $|m - n|_\ell \leq \ell^{-N}$, then $||P_m(T) - P_n(T)||_\ell \leq \ell^{-(N+1)}$, which completes the proof. □

Since $\mathbb{N}$ is dense in $\mathbb{Z}_\ell$, the function f can be uniquely extended to a continuous function

$$\mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell[\![T]\!]$$

which we denote by the same symbol f. If $a \in \mathbb{Z}_\ell$, then we let

$$P_a(T) = f(a) \in \mathbb{Z}_\ell[\![T]\!]. \tag{6}$$

From now on, if

$$a = \sum_{k=0}^{\infty} a_k \ell^k \in \mathbb{Z}_\ell,\ a_k \in \{0, 1, \ldots, \ell - 1\},$$

is an ℓ-adic integer, then we will simplify the notation by writing

$$a = a_0.a_1a_2\ldots;$$

that is, we are using the "carrying to the right" convention.

As an example of an approximated power series $P_a(T)$ from (6) above, let $\ell = 5$ and $a = 1/3 \in \mathbb{Z}_5$. Then

$$\frac{1}{3} = 2.31313131\ldots \in \mathbb{Z}_5,$$

and the sequence of partial sums

$$S_1 = 2,\ S_2 = 17,\ S_3 = 42,\ S_4 = 417,\ S_5 = 1042, \ldots$$

converges to $1/3$ in $\mathbb{Z}_5$. From Proposition 2.1, we have $d_1(n) = n^2$ for all $n \in \mathbb{N}$. Therefore, we have

$$d_1(S_1) = 4,\ d_1(S_2) = 4.212,\ d_1(S_3) = 4.2042,\ d_1(S_4) = 4.2013012,\ d_1(S_5) = 4.20122432, \ldots \in \mathbb{Z}_5.$$

We can see empirically that this sequence seems to stabilize to a number in $\mathbb{Z}_5$. Here it converges to $1/9 \in \mathbb{Z}_5$, since the function $\mathbb{Z}_5 \longrightarrow \mathbb{Z}_5$ defined via $x \mapsto x^2$ is continuous for the 5-adic topology. Therefore

$$\frac{1}{9} = 4.2012\ldots \in \mathbb{Z}_5,$$

is the first coefficient of $P_{1/3}(T)$. A similar analysis can be done for the other coefficients. Note also that by continuity, it follows from the proof of Proposition 3.1 that if $S_n \in \mathbb{N}$ and $a \in \mathbb{Z}_\ell$ satisfy $|S_n - a|_\ell \leq \ell^{-N}$ then $||P_{S_n}(T) - P_a(T)||_\ell \leq \ell^{-(N+1)}$, so that for any coefficient of $P_a(T)$, one can deduce how close S_n has to be to a in order to obtain a desired accuracy. The details are left to the reader, but for the example above, we obtain the following approximation of $P_{1/3}(T)$:

$$P_{1/3}(T) = (4.2012\ldots)T + (4.2342\ldots)T^2 + (2.2130\ldots)T^3 + (0.3400\ldots)T^4 + \ldots \in \mathbb{Z}_5[\![T]\!].$$

It is worth pointing out that since the function $x \mapsto x^2$ is continuous in the ℓ-adic topology, then the power series $P_a(T)$ for any $a \in \mathbb{Z}_\ell$ begins as follows

$$P_a(T) = a^2 T + \ldots \tag{7}$$

Consider now the field

$$\mathbb{Q}(\zeta_{\ell^\infty}) = \mathbb{Q}(\zeta_{\ell^i} \,|\, i = 1, 2, \ldots) \subseteq \mathbb{C}.$$

It is an infinite algebraic extension of $\mathbb{Q}$. From now on, we fix an embedding

$$\tau : \mathbb{Q}(\zeta_{\ell^\infty}) \hookrightarrow \overline{\mathbb{Q}}_\ell.$$

For $i = 0, 1, 2, \ldots$, we let

$$\xi_{\ell^i} = \tau(\zeta_{\ell^i}) \in \overline{\mathbb{Q}}_\ell.$$

Furthermore, if $a \in \mathbb{Z}$, we let

$$\eta_{\ell^i}(a) = \tau(\varepsilon_{\ell^i}(a)) = (1 - \xi_{\ell^i}^a)(1 - \xi_{\ell^i}^{-a}) \in \overline{\mathbb{Q}}_\ell,$$

and we write η_{ℓ^i} instead of $\eta_{\ell^i}(1)$. Recall that the valuation ord_ℓ as well as the absolute value $|\cdot|_\ell$ on $\mathbb{Q}$ can be extended uniquely to $\overline{\mathbb{Q}}_\ell$ and also all the way up to a fixed completion $\mathbb{C}_\ell$ of $\overline{\mathbb{Q}}_\ell$. We will denote the valuation on $\mathbb{C}_\ell$ by v_ℓ and the absolute value by $|\cdot|_\ell$. They are related to one another via $v_\ell(x) = -\log_\ell(|x|_\ell)$ for all $x \in \mathbb{C}_\ell$.

If $x \in \mathbb{Q}(\zeta_{\ell^i})$ and if $\mathcal{L}_i$ is the unique prime ideal of $\mathbb{Q}(\zeta_{\ell^i})$ lying above ℓ, we have

$$v_\ell(\tau(x)) = \frac{1}{\varphi(\ell^i)} \text{ord}_{\mathcal{L}_i}(x), \tag{8}$$

for all $x \in \mathbb{Q}(\zeta_{\ell^i})$. Here φ denotes the Euler φ-function and $\text{ord}_{\mathcal{L}_i}$ is the valuation on $\mathbb{Q}(\zeta_{\ell^i})$ associated to the prime ideal $\mathcal{L}_i$. Since $\text{ord}_{\mathcal{L}_i}(1 - \zeta_{\ell^i}) = 1$, we have

$$v_\ell(\eta_{\ell^i}) = \frac{2}{\varphi(\ell^i)} \text{ and } |\eta_{\ell^i}|_\ell = \ell^{-2/\varphi(\ell^i)}. \tag{9}$$

From now on, we let

$$D = \{x \in \mathbb{C}_\ell \,:\, |x|_\ell < 1\}.$$

Fix $i \in \mathbb{N}$, and let $\alpha = \xi_{\ell^i} - 1$. We have $|\alpha|_\ell = \ell^{-1/\varphi(\ell^i)} < 1$, so that $\alpha \in D$ and thus $|\alpha|_\ell^n \to 0$ as $n \to \infty$. By the theory of Mahler series (see Chapter 4 of [5]), the function $\mathbb{N} \longrightarrow \mathbb{C}_\ell$ defined via

$$a \longrightarrow \xi_{\ell^i}^a$$

extends to a continuous function $\mathbb{Z}_\ell \longrightarrow \mathbb{C}_\ell$ which we denote by the same symbol. Note that if

$$a = \sum_{k=0}^{\infty} a_k \ell^k \in \mathbb{Z}_\ell,\ a_k \in \{0, 1, \ldots, \ell - 1\},$$

then

$$\xi_{\ell^i}^a = \xi_{\ell^i}^{\sum_{k=0}^{i-1} a_k \ell^k}. \tag{10}$$

We obtain for each $i \in \mathbb{N}$ a continuous function $\mathbb{Z}_\ell \longrightarrow \mathbb{C}_\ell$ defined via

$$a \mapsto \eta_{\ell^i}(a) = (1 - \xi_{\ell^i}^a)(1 - \xi_{\ell^i}^{-a}).$$

From (10), we actually have $\eta_{\ell^i}(\mathbb{Z}_\ell) \subseteq \overline{\mathbb{Q}}_\ell$.

Given any $Q(T) = \sum_{k=0}^{\infty} a_k T^k \in \mathbb{Z}_\ell[\![T]\!]$, it defines a continuous function $Q : D \longrightarrow \mathbb{C}_\ell$ via

$$x \mapsto Q(x) = \sum_{k=0}^{\infty} a_k x^k.$$

By (9), we thus get for every $a \in \mathbb{Z}_\ell$ a well-defined number $P_a(\eta_{\ell^i}) \in \mathbb{C}_\ell$.

Lemma 3.2. *For each $i \in \mathbb{N}$, the function $\mathrm{ev}_{\eta_{\ell^i}} : \mathbb{Z}_\ell[\![T]\!] \longrightarrow \mathbb{C}_\ell$ given by*

$$Q(T) \mapsto \mathrm{ev}_{\eta_{\ell^i}}(Q(T)) = Q(\eta_{\ell^i})$$

is uniformly continuous.

Proof. It suffices to show that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $Q(T) \in \mathfrak{m}^N$, then $|Q(\eta_{\ell^i})|_\ell < \varepsilon$. To simplify the notation, let $x = |\eta_{\ell^i}|_\ell = \ell^{-2/\varphi(\ell^i)} < 1$. If

$$Q(T) = \sum_{k=0}^{\infty} a_k T^k \in \mathfrak{m}^N,$$

then $|a_k|_\ell \leq \ell^{k-N}$ for $k = 0, \ldots, N$, and thus

$$\begin{aligned} \Big| \sum_{k=0}^{\infty} a_k (\eta_{\ell^i})^k \Big|_\ell &\leq \sum_{k=0}^{N-1} |a_k|_\ell \cdot x^k + \sum_{k=N}^{\infty} x^k \\ &\leq \frac{1}{\ell^N} \sum_{k=0}^{N-1} (\ell x)^k + \frac{x^N}{1-x}, \end{aligned}$$

which can be made arbitrarily small for N large. □

As a consequence, we obtain the following result.

Corollary 3.3. *Given $a \in \mathbb{Z}_\ell$ and any $i \in \mathbb{Z}_{\geq 0}$, we have $P_a(\eta_{\ell^i}) = \eta_{\ell^i}(a)$.*

Proof. If $i = 0$, the equality is clear. If $i \geq 1$, then the function $\mathrm{ev}_{\eta_{\ell^i}} \circ f : \mathbb{Z}_\ell \longrightarrow \mathbb{C}_\ell$ is continuous by Lemma 3.2. Furthermore, the function $\mathbb{Z}_\ell \longrightarrow \mathbb{C}_\ell$ given by $a \mapsto \eta_{\ell^i}(a)$ is also continuous as we pointed out before. Since both these functions agree on $\mathbb{N}$ by (1) and $\mathbb{N}$ is dense in $\mathbb{Z}_\ell$, the claim follows. □

4. Abelian ℓ-towers of bouquets

For this section, we assume that the reader is familiar with [9] and in particular with the notion of an abelian ℓ-tower of multigraphs. (See Definition 4.1 of [9].) Recall that if S is a finite set and

$$i : S \longrightarrow \mathbb{Z}_\ell$$

is any function for which there exists $s \in S$ such that $i(s) \in \mathbb{Z}_\ell^\times$, then one gets a regular abelian ℓ-tower of connected multigraphs

$$X = B_{|S|} \longleftarrow X(\mathbb{Z}/\ell\mathbb{Z}, S, i_1) \longleftarrow X(\mathbb{Z}/\ell^2\mathbb{Z}, S, i_2) \longleftarrow \ldots \longleftarrow X(\mathbb{Z}/\ell^n\mathbb{Z}, S, i_n) \longleftarrow \ldots,$$

where B_t denotes a bouquet with t loops and $X(\mathbb{Z}/\ell^n\mathbb{Z}, S, i_n)$ is the Cayley-Serre multigraph associated to the data $(\mathbb{Z}/\ell^n\mathbb{Z}, S, i_n)$. The function i_n is the one obtained from the composition

$$S \xrightarrow{i} \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell \xrightarrow{\simeq} \mathbb{Z}/\ell^n\mathbb{Z}.$$

Theorem 5.6 of [9] applies to regular abelian ℓ-towers as above in the case where

$$i(S) \subseteq \mathbb{Z}.$$

We can now remove this condition. (The case $|S| = 1$ has already been treated separately. See the discussion after Definition 4.1 of [9].)

Theorem 4.1. *Let $S = \{s_1, \ldots, s_t\}$ be a finite set with cardinality $t \geq 2$, and let $i : S \longrightarrow \mathbb{Z}_\ell$ be a function. For $j = 1, \ldots, t$, let a_j be the ℓ-adic integer satisfying $i(s_j) = a_j$. Assume that at least one of $a_1, \ldots, a_t$ is relatively prime with ℓ (so that our Cayley-Serre multigraphs are connected). Consider the regular abelian ℓ-tower*

$$X = B_t \longleftarrow X(\mathbb{Z}/\ell\mathbb{Z}, S, i_1) \longleftarrow X(\mathbb{Z}/\ell^2\mathbb{Z}, S, i_2) \longleftarrow \ldots \longleftarrow X(\mathbb{Z}/\ell^n\mathbb{Z}, S, i_n) \longleftarrow \ldots$$

and define the ℓ-adic integers c_j via

$$\begin{aligned} Q(T) &= P_{a_1}(T) + \ldots + P_{a_t}(T) \\ &= c_1 T + c_2 T^2 + \ldots \in \mathbb{Z}_\ell[\![T]\!]. \end{aligned}$$

Let

$$\mu = \min\{v_\ell(c_j) \mid j = 1, 2, \ldots\},$$

and

$$\lambda = \min\{2j \mid j \in \mathbb{N} \text{ and } v_\ell(c_j) = \mu\} - 1.$$

If κ_n denotes the number of spanning trees of $X(\mathbb{Z}/\ell^n\mathbb{Z}, S, i_n)$, and if n_0 is an integer satisfying

$$n_0 \geq \log_\ell\left(\frac{\ell}{\ell - 1}(\lambda + 1)\right),$$

then there exists a constant $\nu \in \mathbb{Z}$ (depending also on the a_j) such that

$$\mathrm{ord}_\ell(\kappa_n) = \mu\ell^n + \lambda n + \nu,$$

when $n \geq n_0$.

Proof. Using (8), the first equation in the proof of Theorem 5.6 in [9] becomes

$$\mathrm{ord}_\ell(\kappa_n) = -n + \sum_{i=1}^{n} \varphi(\ell^i) v_\ell(\eta_{\ell^i}(a_1) + \ldots + \eta_{\ell^i}(a_t)).$$

By Corollary 3.3, we have

$$\begin{aligned} \eta_{\ell^i}(a_1) + \ldots + \eta_{\ell^i}(a_t) &= P_{a_1}(\eta_{\ell^i}) + \ldots + P_{a_t}(\eta_{\ell^i}) \\ &= Q(\eta_{\ell^i}). \end{aligned}$$

We claim that for i large

$$v_\ell\left(Q(\eta_{\ell^i})\right) = \mu + \frac{\lambda + 1}{\varphi(\ell^i)}. \tag{11}$$

Let us write $Q(T) = \ell^\mu \cdot R(T)$ for some

$$R(T) = \sum_{k=1}^{\infty} e_k T^k \in \mathbb{Z}_\ell[\![T]\!] \smallsetminus \ell\mathbb{Z}_\ell[\![T]\!].$$

To prove (11), it suffices to show

$$v_\ell\left(R(\eta_{\ell^i})\right) = \frac{\lambda + 1}{\varphi(\ell^i)}, \tag{12}$$

when i is large. Let us define

$$k_0 = \min\{k \in \mathbb{N} \mid v_\ell(e_k) = 0\}$$

so that we have $\lambda + 1 = 2k_0$. We assume i is large enough so that

$$\frac{2k_0}{\varphi(\ell^i)} \leq 1 \tag{13}$$

as this ensures that the values

$$v_\ell(e_k(\eta_{\ell^i})^k) = v_\ell(e_k) + \frac{2k}{\varphi(\ell^i)}$$

are distinct for $k = 1, \ldots, k_0$. (Indeed, if two such values were equal with $1 \leq j < k \leq k_0$, then $2(k-j)/\varphi(\ell^i) \in \mathbb{Z}$ and hence $\varphi(\ell^i) \leq 2(k-j) < 2k_0$.) From this, together with the fact that for $k < k_0$ one has

$$v_\ell(e_k(\eta_{\ell^i})^k) > v_\ell(e_k) \geq 1\,,$$

we obtain

$$v_\ell\left(\sum_{k=1}^{k_0} e_k(\eta_{\ell^i})^k\right) = \frac{2k_0}{\varphi(\ell^i)} = \frac{\lambda+1}{\varphi(\ell^i)}\,.$$

In addition, we have

$$v_\ell\left(\sum_{k=k_0+1}^{\infty} e_k(\eta_{l^i})^k\right) \geq \min_{k>k_0} \frac{2k}{\varphi(\ell^i)} = \frac{2(k_0+1)}{\varphi(\ell^i)}$$

and (12) follows.

Therefore, by (13), if n_0 is an integer satisfying

$$n_0 \geq \log_\ell\left(\frac{\ell}{\ell-1}(\lambda+1)\right),$$

then there exists an integer C such that if $n \geq n_0$, then

$$\begin{aligned}
\mathrm{ord}_\ell(\kappa_n) &= -n + C + \sum_{i=n_0}^{n} \varphi(\ell^i) v_\ell(Q(\eta_{\ell^i})) \\
&= -n + C + \sum_{i=n_0}^{n} (\mu \cdot \varphi(\ell^i) + (\lambda+1)) \\
&= -n + C + (\lambda+1)(n-(n_0-1)) + \mu(\ell^n - \ell^{n_0-1}),
\end{aligned}$$

and this ends the proof. □

Remark 4.2. We point out that since $c_1 = a_1^2 + \ldots + a_t^2$ by (7), if

$$v_\ell(a_1^2 + \ldots + a_t^2) = 0,$$

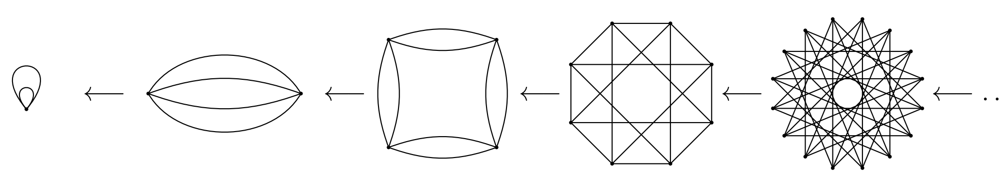
then we have $\mu = 0, \lambda = 1$ and $\nu = 0$.

5. Examples

The computations of the number of spanning trees in this section have been performed with the software [8]. Once a multigraph has been entered in SageMath (we refer the reader to the documentation in [8]), the command `spanning_trees_count` can be used to calculate its number of spanning trees. There are several papers in the literature whose aim is to find a formula for the number of spanning trees for various families of multigraphs, the most well-known one probably being Cayley's formula for the number of spanning trees of a complete graph. The multigraphs arising in abelian ℓ-towers of bouquets are circulant multigraphs. We refer the reader to [3] and the references therein for formulas that have been derived to calculate the number of spanning trees for such graphs.

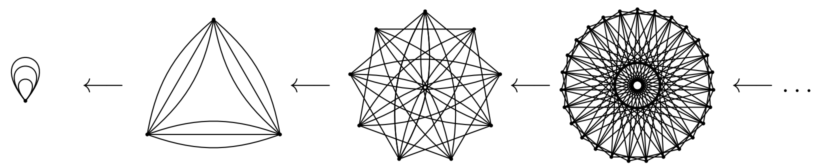
The computations of the approximated power series $P_a(T)$ and $Q(T)$ have been performed with the software [7] as explained in the discussion following Proposition 3.1. Once μ and λ are known, it suffices to calculate a single κ_n for some $n \geq n_0$ in order to obtain ν as well, but we give the first few values of κ_n for expository purposes.

(1) Let $a_1 = 1/3, a_2 = 3/5$ and $\ell = 2$. Then, we get:



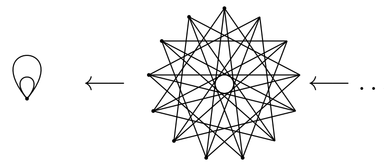
The power series Q starts as follows
$$Q(T) = (0.1010\ldots)T + (0.1000\ldots)T^2 + (1.0101\ldots)T^3 + (0.0000\ldots)T^4 + \ldots \in \mathbb{Z}_2[\![T]\!],$$
so we should have $\mu = 0$ and $\lambda = 5$. We calculate
$$\kappa_0 = 1, \kappa_1 = 2^2, \kappa_2 = 2^5, \kappa_3 = 2^{12}, \kappa_4 = 2^{17} \cdot 17^2, \kappa_5 = 2^{22} \cdot 17^2 \cdot 1217^2, \ldots$$
We have
$$\mathrm{ord}_2(\kappa_n) = 5n - 3,$$
for all $n \geq 3$.

(2) Let $a_1 = 1/2, a_2 = 1/5, a_3 = 1/7$ and $\ell = 3$. Then, we get:



The power series Q starts as follows
$$Q(T) = (0.0111\ldots)T + (1.1020\ldots)T^2 + (1.0200\ldots)T^3 + \ldots \in \mathbb{Z}_3[\![T]\!],$$
so we should have $\mu = 0$ and $\lambda = 3$. We calculate
$$\kappa_0 = 1, \kappa_1 = 3^3, \kappa_2 = 3^6 \cdot 19^2, \kappa_3 = 3^9 \cdot 19^2 \cdot 703459^2, \ldots$$
and we have
$$\mathrm{ord}_3(\kappa_n) = 3n,$$
for all $n \geq 0$.

(3) Let $\ell = 13$ and $a_1 = \sqrt{3} = 4.868\ldots, a_2 = \sqrt{10} = 6.264\ldots \in \mathbb{Z}_{13}$. Then, we get:



The power series Q starts as follows
$$Q(T) = 13T - 8T^2 + (3.4961\ldots)T^3 + \ldots \in \mathbb{Z}_{13}[\![T]\!],$$
so we should have $\mu = 0$ and $\lambda = 3$. We calculate
$$\mathrm{ord}_{13}(\kappa_0) = 0, \mathrm{ord}_{13}(\kappa_1) = 3, \mathrm{ord}_{13}(\kappa_2) = 6, \mathrm{ord}_{13}(\kappa_3) = 9, \ldots$$
We have
$$\mathrm{ord}_{13}(\kappa_n) = 3n,$$
for all $n \geq 0$.

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