# THE RANK ONE STARK CONJECTURE FOR ABELIAN EXTENSIONS OF QUADRATIC IMAGINARY NUMBER FIELDS

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by

William R. Grenard

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DEDICATION

To my parents,

Robert and Rene.

## ACKNOWLEDGEMENTS

It is a common sentiment that worthy accomplishments, academic or otherwise, are never truly achieved alone. This is something I have come to agree with, and I happily take this moment to thank some of those who have helped me reach the place I am today.

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ABSTRACT

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In this thesis, we provide an exposition of the proof for the rank one Stark conjecture for abelian extensions of quadratic imaginary number fields. A proof of this conjecture was first given by Stark in 1980, and in this thesis we expound upon Stark's proof by compiling in a cohesive manner topics that are known in the literature. This is done with the intention of making the proof as accessible as possible. We explain Kronecker's second limit formula, as well as its connection to special values of the L-functions of Stark's conjecture. In addition we explain various properties of the Siegel functions, which ultimately allow one to construct algebraic numbers satisfying special properties. Finally, we explain how all of this knowledge can be used to provide a proof of the abelian rank one Stark conjecture when the base field is a quadratic imaginary number field.

# 1 Introduction

Between 1971 and 1980 Harold Stark wrote a series of four papers discussing the special value $s = 0$ of certain abelian L-functions. In the final paper of the series ([19]) he introduced what is now referred to as Stark's abelian rank one conjecture. This conjecture is concerned with the S-imprimitive L-function associated to an abelian extension $K/k$ of number fields, denoted

$$L_{K/k,S}(s, \chi),$$

where $\chi$ is a character of the Galois group of $K/k$. Stark conjectured that under certain conditions imposed on the set $S$ the value $L'_{K/k,S}(0, \chi)$ takes on a very specific form given in (2.1). Notably, the conjecture predicts the existence of special algebraic numbers known as Stark units which are related via (2.1) to $L'_{K/k,S}(0, \chi)$. In his original paper, Stark proved that his conjecture is true for abelian extensions of quadratic imaginary number fields, and in this thesis we compile what is known from the literature to provide a detailed explanation of the proof with the aim of making it as accessible as possible.

The proof of the conjecture can be separated broadly into two main parts. The first part consists of producing a suitable expression for $L'_{K/k,S}(0, \chi)$ in terms of special values of the Siegel functions. The second consists of verifying that these special values are algebraic numbers which satisfy special properties. In Chapter 3, we complete this first part of the proof. There, we follow [12] to prove Kronecker's second limit formula and use it to produce an expression for $L_{K/k,S}(1, \chi)$. Then we apply a known functional equation to produce an equivalent expression at $s = 0$ in terms of special values of the Siegel functions.

In Chapter 4 we lay the groundwork for showing that the special values of the

Siegel functions are actually Stark units. Following [10], we prove that under suitable assumptions, the Siegel functions are automorphic functions on $\Gamma(N)$ and that together they define a homogeneous function on enhanced elliptic curves for $\Gamma_1(N)$. Moreover, we also prove that the Siegel functions form a Fricke family.

In Chapter 5 we define the Siegel-Ramachandra invariant as in [10] and [22], which is a ray class invariant arising from the Siegel functions. We use the relationship between the Siegel functions and the corresponding function on enhanced elliptic curves for $\Gamma_1(N)$ to show that the special values of the Siegel functions obtained in the expression for $L'_{K/k,S}(0,\chi)$ are indeed the Siegel-Ramachandra invariants. The results of Chapter 4 are then employed along with the Shimura reciprocity law and other known results from the theory of complex multiplication on elliptic curves to prove various properties of the Siegel-Ramachandra invariant. The details of how the theory of CM is used in this application are not included, but clear references for this part of the theory are provided.

At last, we show how the properties of the Siegel-Ramachandra invariant imply that Stark's abelian rank one conjecture is true for abelian extensions of quadratic imaginary number fields.

## 1.1 Notation

Most of the notation used throughout will be explained when it is introduced. However, what follows is a list of important notation and terminology for quick access.

1. Unless otherwise specified $K$ is a number field, and its ring of integers is denoted by $\mathcal{O}_K$ or $\mathcal{O}$ when the field is understood. An integral ideal of $\mathcal{O}_K$ is often referred to as an integral ideal of $K$. Given an integral ideal $\mathfrak{a}$ of $K$ let $\mathbb{N}(\mathfrak{a}) := |\mathcal{O}_K/\mathfrak{a}|$ denote the norm of $\mathfrak{a}$.

2. The discriminant of a fractional ideal $\mathfrak{a}$ of $K$ is denoted by $d(\mathfrak{a})$ and the discriminant of $K$ itself is denoted $d_K$.

3. Given an extension of number fields $K/k$, the relative different ideal $\mathfrak{d} = \mathfrak{d}_{K/k}$

is defined by the relationship

$$\mathfrak{d}^{-1} = \{\alpha \in K \mid \mathrm{Tr}_{K/k}(\alpha\mathcal{O}_K) \subseteq \mathbb{Z}\}.$$

4. Given an extension $K/k$ of number fields, the set of infinite places of $k$ is denoted $S_\infty$, and the set of finite places of $k$ which ramify in $K/k$ is denoted $S_{ram}$. If $S$ is a set of places of $k$, the set of places of $K$ lying above the places in $S$ is denoted by $S_K$.

5. Any infinite place corresponding to a pair of complex embeddings of $k$ is said to split completely in $K/k$. An infinite place corresponding to a real embedding of $k$ only splits completely if it does *not* extend to a complex embedding in $K$.

6. Let $S$ be a set of places of a number field $k$ containing the infinite places. An $S$-unit is an element $\varepsilon \in \mathcal{O}_k$ for which $|\varepsilon|_v = 1$ for all places $v$ of $k$ that are not in $S$. The $S$-units form a subgroup of $k^\times$. When $S = S_\infty$ the group of $S$-units is precisely the group of units of $\mathcal{O}_k$. Furthermore, if $K/k$ is a finite extension, the $S_K$-units in $K$ will sometimes be referred to simply as $S$-units in $K$ if there is no danger of confusion.

7. Let $k$ be a number field and $\mathfrak{m}$ an integral ideal of $k$. Then $k_{\mathfrak{m}}^{res}$ denotes the ray class field of $k$ modulo $\mathfrak{m}$ (in the restricted sense). Furthermore, $G_{\mathfrak{m}}^{res}$ denotes the Galois group of the extension $k_{\mathfrak{m}}^{res}/k$. When $k$ is quadratic imaginary we simply write $k_{\mathfrak{m}}$ and $G_{\mathfrak{m}}$, since in this case the ray class field in the restricted sense is equal to the ray class field in the non-restricted sense.

8. The complex upper half plane is denoted by $\mathbb{H}$. That is $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

## 1.2 The Frobenius Automorphism

Let $p$ be a rational prime and consider the finite field $\mathbb{F}_{p^n}$ with $p^n$ elements. Recall that $\mathbb{F}_{p^n}$ is Galois over its prime subfield $\mathbb{F}_p$ and that $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic. It is generated by the automorphism $\sigma_p : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ defined by $x \mapsto x^p$. This map $\sigma_p$ is called the Frobenius automorphism or the Frobenius map.

More generally, any finite extension of a finite field takes the form $\mathbb{F}_{p^n}/\mathbb{F}_{p^m}$ for some prime $p$ and some $m, n \in \mathbb{N}$ with $m \mid n$. Moreover, $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_{p^m}$ and $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m})$ is cyclic and generated by $\sigma_p^m$. We will refer to this map as the $p^m$-power map. For proofs of these facts see [4, Ch. 11.1].

Now, let $K/k$ be an abelian extension of number fields with Galois group $G$. Let $\mathfrak{p}$ be a prime ideal of $k$ and $\mathfrak{P}$ a prime ideal of $K$ lying above $\mathfrak{p}$. We consider the residue fields $\mathbb{F}_{\mathfrak{P}} := \mathcal{O}_K/\mathfrak{P}$ and $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_k/\mathfrak{p}$. These are indeed fields because $\mathfrak{P}$ and $\mathfrak{p}$ are prime ideals in the Dedekind domains $\mathcal{O}_K$ and $\mathcal{O}_k$ respectively. Every prime ideal in a Dedekind domain is maximal, by definition, so $\mathfrak{P}$ and $\mathfrak{p}$ are in fact maximal. Moreover, as the notation implies, both $\mathbb{F}_{\mathfrak{P}}$ and $\mathbb{F}_{\mathfrak{p}}$ are finite. In fact we have $|\mathbb{F}_{\mathfrak{P}}| = \mathbb{N}(\mathfrak{P})$ and $|\mathbb{F}_{\mathfrak{p}}| = \mathbb{N}(\mathfrak{p})$.

Finally, note that $\mathbb{F}_{\mathfrak{P}}$ is a field extension of $\mathbb{F}_{\mathfrak{p}}$. This is clear because $\alpha+\mathfrak{p} \mapsto \alpha+\mathfrak{P}$ is a well defined injective ring morphism since $\mathfrak{P}\cap\mathcal{O}_k = \mathfrak{p}$. Thus, $\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ is cyclic and generated by the $\mathbb{N}(\mathfrak{p})$-power map $\alpha + \mathfrak{P} \mapsto \alpha^{\mathbb{N}(\mathfrak{p})} + \mathfrak{P}$, which is some power of the Frobenius automorphism on $\mathbb{F}_{\mathfrak{P}}$.

We now discuss how to use these results to associate a Frobenius automorphism to an unramified prime ideal in a Galois extension of number fields. The main idea is that when $\mathfrak{p}$ is unramified, the Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ can be lifted to a unique element $\sigma_{\mathfrak{p}} \in G$ which we consider as the Frobenius map associated to $\mathfrak{p}$. For a more detailed discussion of what follows see [14, Ch. 1.9].

Using the same notation as above, it can be shown that the Galois group $G$ of $K/k$ acts transitively on the set of prime ideals above $\mathfrak{p}$. Thus, if $\mathfrak{P}$ is a prime lying above $\mathfrak{p}$ then for each $\sigma \in G$ we have that $\sigma\mathfrak{P}$ is also a prime lying above $\mathfrak{p}$. Moreover, if $\mathfrak{P}'$ is another prime lying above $\mathfrak{p}$ then there exists a $\tau \in G$ such that $\tau\mathfrak{P} = \mathfrak{P}'$. The stabilizer subgroup of $G$ with respect to a given prime $\mathfrak{P}$ is denoted

$$G_{\mathfrak{P}} = \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\},$$

and is called the decomposition group of $\mathfrak{P}$ for the extension $K/k$. Each $\sigma \in G_{\mathfrak{P}}$ induces an automorphism $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ defined by

$$\tilde{\sigma}(\alpha + \mathfrak{P}) = \sigma(\alpha) + \mathfrak{P}.$$

It is simple to check that $\tilde{\sigma}$ is well defined as long as $\sigma \in G_{\mathfrak{P}}$. Thus, we have defined a map

$$\begin{aligned} \varphi_{\mathfrak{P}} : G_{\mathfrak{P}} &\to \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) \\ \sigma &\mapsto \tilde{\sigma} \end{aligned}$$

One can show that $\varphi_{\mathfrak{P}}$ is surjective. Its kernel

$$I_{\mathfrak{P}} = \{\sigma \in G_{\mathfrak{P}} \mid \sigma(\alpha) \equiv \alpha \pmod{\mathfrak{P}}, \text{ for all } \alpha \in \mathcal{O}_K\}$$

is called the inertia subgroup of $\mathfrak{P}$ for the extension $K/k$. Thus, we have the following short exact sequence

$$1 \longrightarrow I_{\mathfrak{P}} \longrightarrow G_{\mathfrak{P}} \xrightarrow{\varphi_{\mathfrak{P}}} \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) \longrightarrow 1.$$

Note that since $K/k$ is Galois, all primes in $K$ lying above $\mathfrak{p}$ have the same ramification index $e$. It can be shown that $|I_{\mathfrak{P}}| = e$. We now assume that $\mathfrak{p}$ is unramified in $K$. Then $|I_{\mathfrak{P}}| = 1$ and so for each prime $\mathfrak{P}$ lying above $\mathfrak{p}$ we have an isomorphism

$$\varphi_{\mathfrak{P}} : G_{\mathfrak{P}} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}).$$

As discussed above, $\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ is generated by the $|\mathbb{F}_{\mathfrak{p}}|$-power map $\alpha+\mathfrak{P} \mapsto \alpha^{|\mathbb{F}_{\mathfrak{p}}|}+\mathfrak{P}$. The lift of this map via the isomorphism $\varphi_{\mathfrak{P}}$ is called the Frobenius automorphism associated to $\mathfrak{P}$. It is denoted by $(K/k, \mathfrak{P})$ or by $\sigma_{\mathfrak{P}}$ if the extension is understood. This map has a simple characterization. Namely, it is the unique $\sigma \in G$ such that

$$\sigma(\alpha) \equiv \alpha^{\mathbb{N}(\mathfrak{p})} \bmod \mathfrak{P}$$

for all $\alpha \in \mathcal{O}_K$. Indeed, it is clear by construction that $\sigma_{\mathfrak{P}}$ has this property. To see why it is unique, let $x \in \mathfrak{P}$ and $\sigma \in G$ be a map with the given property. Thus, we have

$$\sigma(x) \equiv x^{\mathbb{N}(\mathfrak{p})} \equiv 0 \bmod \mathfrak{P}.$$

Since this holds for any $x \in \mathfrak{P}$ we have that $\sigma\mathfrak{P} \subseteq \mathfrak{P}$. But $\sigma\mathfrak{P}$ is a prime ideal in the Dedekind domain $\mathcal{O}_K$, and thus it is maximal. So in fact $\sigma\mathfrak{P} = \mathfrak{P}$. This means that $\sigma \in G_{\mathfrak{P}}$ and the isomorphism $\varphi_{\mathfrak{P}}$ sends $\sigma$ to the $|\mathbb{F}_{\mathfrak{p}}|$-power map. Since $\varphi_{\mathfrak{P}}$ is bijective, there is only one such $\sigma$.

We have succeeded in associating a unique automorphism in $G$, which we call the Frobenius automorphism, to each prime $\mathfrak{P}$ lying above an unramified prime $\mathfrak{p}$. In the case where $G$ is abelian we can simplify the situation further, because in this case the map $\sigma_{\mathfrak{P}}$ is in fact independent of $\mathfrak{P}$ and only depends on $\mathfrak{p}$. To see why, let $\mathfrak{P}'$ be another prime lying above $\mathfrak{p}$. Then

$$\sigma_{\mathfrak{P}'}(\alpha) \equiv \alpha^{\mathbb{N}(\mathfrak{p})} \bmod \mathfrak{P}'$$

for all $\alpha \in \mathcal{O}_K$. Let $\tau \in G$ such that $\mathfrak{P}' = \tau\mathfrak{P}$. Such a $\tau$ can be found since $G$ acts transitively on the set of primes above $\mathfrak{p}$. Note, if $\alpha \in \mathcal{O}_K$ then also $\tau(\alpha) \in \mathcal{O}_K$. So

$$\begin{aligned} \sigma_{\mathfrak{P}'}(\tau(\alpha)) - (\tau(\alpha))^{\mathbb{N}(\mathfrak{p})} &\in \tau\mathfrak{P} \\ \tau(\sigma_{\mathfrak{P}'}(\alpha) - \alpha^{\mathbb{N}(\mathfrak{p})}) &\in \tau\mathfrak{P} \\ \sigma_{\mathfrak{P}'}(\alpha) - \alpha^{\mathbb{N}(\mathfrak{p})} &\in \mathfrak{P} \end{aligned}$$

for all $\alpha \in \mathcal{O}_K$, where the last step follows because $\tau$ is injective. Rewriting this result using the notation above, we have

$$\sigma_{\mathfrak{P}'}(\alpha) \equiv \alpha^{\mathbb{N}(\mathfrak{p})} \bmod \mathfrak{P}$$

for all $\alpha \in \mathcal{O}_K$. Thus, $\sigma_{\mathfrak{P}'} = \sigma_{\mathfrak{P}}$, since $\sigma_{\mathfrak{P}}$ is the unique element of $G$ which satisfies this relationship. This shows that when $K/k$ is an abelian Galois extension of number fields we may speak of *the* Frobenius automorphism of an unramified prime ideal $\mathfrak{p}$ of $k$ and denote it by $\sigma_{\mathfrak{p}}$.

## 1.3 Valuations and Absolute Values

**Definition 1.1.** A valuation on a field $K$ is a function $v : K \to \mathbb{Z} \cup \{\infty\}$ satisfying the following properties:

(1) $v(x) = \infty$ if and only if $x = 0$,

(2) $v(xy) = v(x) + v(y)$ for all $x, y \in K$,

(3) $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

A class of valuations central to algebraic number theory are the $\mathfrak{p}$-adic valuations, which we now recall. Let $K$ be a number field. Then for any $\alpha \in K^\times$ and any prime ideal $\mathfrak{p}$ of $K$ we may write $\alpha\mathcal{O}_K = \mathfrak{p}^n\mathfrak{a}$ where $\mathfrak{a}$ is a fractional ideal such that $(\mathfrak{a}, \mathfrak{p}) = 1$ and $n \in \mathbb{Z}$ is unique. It can be verified that the function $\mathrm{ord}_\mathfrak{p} : K \to \mathbb{Z} \cup \{\infty\}$ defined by $\mathrm{ord}_\mathfrak{p}(\alpha) = n$ for $\alpha \neq 0$ and $\mathrm{ord}_\mathfrak{p}(0) = \infty$ defines a valuation on $K$.

**Definition 1.2.** An absolute value on a field $K$ is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

(1) $|x| = 0$ if and only if $x = 0$,

(2) $|xy| = |x| \cdot |y|$ for all $x, y \in K$,

(3) $|x+y| \leq |x| + |y|$ for all $x, y \in K$.

First note that the map which sends every nonzero element of $K$ to $1 \in \mathbb{R}$ and sends $0 \in K$ to $0 \in \mathbb{R}$ is an absolute value on $K$. This is referred to as the trivial absolute value. All absolute values from this point forward are assumed to be non-trivial.

Consider an arbitrary absolute value $|\cdot|$ on $K$. Letting $x = y = 1$ in property (2) of Definition 1.2 implies that $|1_K| = 1$. Thus, if $\zeta$ is an $n$-th root of unity in $K$ it follows that

$$1 = |1_K| = |\zeta^n| = |\zeta|^n,$$

where the last equality follows by applying property (2) repeatedly. Since $|\zeta| \in \mathbb{R}_{\geq 0}$ this implies that $|\zeta| = 1$.

We'll now discuss two ways to construct absolute values on a field $K$. For number fields, the absolute values produced by these two methods turn out to be the only nontrivial absolute values up to equivalence of absolute values (see Theorem 1.3).

Firstly, a class of absolute values arises from embeddings of $K$ into $\mathbb{C}$. Indeed, if $\tau : K \to \mathbb{C}$ is an embedding, the function defined by $|x|_\tau = |\tau(x)|$ is verified to be an absolute value on $K$. Here $|\cdot|$ refers to the standard absolute value on $\mathbb{C}$.

Another class of absolute values arise from valuations on $K$. Given a valuation $v$ on $K$ and a real constant $a > 1$ the function defined by $|x| = a^{-v(x)}$ for all $x \in K^\times$ and $|0| = 0$ defines an absolute value on $K$. The first two properties of the absolute value follow from the defining properties of $v$. The triangle inequality is implied by the following calculation. Let $x, y \in K^\times$ and recall that $a > 1$. We have

$$\begin{aligned} |x+y| &= a^{-v(x+y)} \\ &\leq a^{-\min\{v(x),v(y)\}} \\ &= a^{\max\{-v(x),-v(y)\}} \\ &= \max\{|x|,|y|\}. \end{aligned} \tag{1.1}$$

It is clear that this inequality still holds if one or both of $x$ and $y$ are zero, so it is in fact true for all $x, y \in K$. Since $\max\{|x|,|y|\} \leq |x| + |y|$ this is stronger than the triangle inequality. An absolute value which satisfies (1.1) is called a non-archimedean absolute value. All others are called archimedean absolute values.

Absolute values on a field can be classified nicely up to a certain equivalence relation, described as follows. An absolute value endows the field $K$ with the structure of a metric space via the distance function $d(x, y) = |x - y|$. In particular, a field with an absolute value is a topological space, and two absolute values are said to be equivalent if they induce the same topology on $K$. It can be shown that absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if and only if there exists a constant $s > 0$ such that $|x|_1 = |x|_2^s$ for all $x \in K$.

For archimedean valuations defined via embeddings, each real embedding defines a distinct absolute value on $K$. That is, if $\tau_1$ and $\tau_2$ are distinct real embeddings then $|\cdot|_{\tau_1}$ and $|\cdot|_{\tau_2}$ are non-equivalent absolute values. Furthermore, each pair of complex conjugate non-real embeddings defines a distinct absolute value on $K$. That is, if $\sigma_1$ and $\sigma_2$ are non-real embeddings then the absolute values $|\cdot|_{\sigma_1}$ and $|\cdot|_{\sigma_2}$ are equivalent if and only if $\sigma_2 = \overline{\sigma}_1$.

For non-archimedean valuations defined as above by $|x|_a = a^{-v(x)}$ the choice of $a$ has no effect on the topology induced by the absolute value. Indeed, if $|x|_b = b^{-v(x)}$ for some $b > 1$, then we may choose $s > 0$ such that $a = b^s$. In which case

$$|x|_a = a^{-v(x)} = (b^s)^{-v(x)} = (b^{-v(x)})^s = |x|_b^s,$$

so the absolute values $|\cdot|_a$ and $|\cdot|_b$ are equivalent.

An important absolute value which arises from this construction is the $\mathfrak{p}$-adic absolute value on a number field $K$. This is defined for any prime ideal $\mathfrak{p}$ of $K$ by

$$|x|_{\mathfrak{p}} = \mathbb{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}$$

for all $x \in K$. Based on the the preceding discussion, the base of the expression on the right hand side could be any real number greater than 1 and the resulting absolute value would be equivalent to the one written here. Using a base of $\mathbb{N}(\mathfrak{p})$ is particularly convenient, however, because it results in the product formula (1.2) to be discussed shortly.

The following theorem states that for a number field $K$, the equivalence classes of absolute values on $K$ which have been described above are in fact the only ones.

**Theorem 1.3** (Ostrowski)**.** *Let $K$ be a number field. A non-archimedean absolute value on $K$ is equivalent to $|\cdot|_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of $K$. An archimedean absolute value on $K$ is equivalent to $|\cdot|_{\tau}$ for some embedding $\tau : K \to \mathbb{C}$.*

*Proof.* See [2, Ch. 4.4]. □

Finally, we describe some terminology and notation which nicely packages the ideas of the previous discussion and explains standard conventions. An equivalence class of absolute values on $K$ is called a place of $K$. Extending the terminology used for absolute values, we say that a place is non-archimedean if its representative absolute values are non-archimedian. Otherwise, we say that the place is archimedean. It is also common to refer to the archimedean places as infinite places, and the non-archimedian places as finite places, or primes. Often, places are denoted with lowercase letters such as $v$ and $w$.

For each place $v$ a representative absolute value denoted by $|\cdot|_v$ is chosen as follows.

(1) If $v$ is a non-archimedean absolute value corresponding to the prime ideal $\mathfrak{p}$ then $|x|_v := |x|_{\mathfrak{p}} = \mathbb{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}$.

(2) If $v$ is an archimedean absolute value corresponding to a real embedding $\tau$ then $|x|_v := |x|_{\tau} = |\tau(x)|$.

(3) If $v$ is an archimedean absolute value corresponding to a pair of complex conjugate embeddings $\{\sigma, \overline{\sigma}\}$ then $|x|_v := |x|_{\sigma}^2 = |\sigma(x)|^2$.

Two choices are made here, by convention, that are perhaps not immediately obvious. The first is choosing $\mathbb{N}(\mathfrak{p})$ for the base in (1), and the second is the power of 2 on the map $x \mapsto |x|^2$ in (3). Both of these choices are made to ensure that the following product formula holds for all $\alpha \in K^\times$:

$$\prod_v |\alpha|_v = 1. \tag{1.2}$$

If $L/K$ is an extension of number fields, and $v$ is a place of $K$, we say that a place $w$ of $L$ lies above $v$ if the absolute value $|\cdot|_w$ restricted to $K$ is equivalent to the absolute value $|\cdot|_v$. In this case, we write $w \mid v$. If $v$ is finite place it is said to split completely in $L/K$ if its corresponding prime ideal splits completely. If, rather, $v$ is an infinite place corresponding to a real embedding $\tau$, and $\tau$ extends to a complex embedding of $L$, then $v$ is said to ramify. Otherwise, $v$ is said to split completely.

It must be noted that the function $x \mapsto |x|_\sigma^2$ defined for a non-real embedding $\sigma$ is not strictly an absolute value because it fails to satisfy the triangle inequality. For this reason, some authors will follow Artin and replace the triangle inequality (property (3) in Definition 1.2) with the following weaker requirement:

(3′) There exists a $c \in \mathbb{R}$ such that $|x| \leq 1$ implies $|x+1| \leq c$.

It is straightforward to verify that the absolute values associated to pairs of complex conjugate embeddings satisfy (3′) for $c = 4$. Moreover, one can also check that the triangle inequality implies (3′) with $c = 2$, so the absolute values associated to primes and real embeddings satisfy (3′).

Whether one chooses to use (3) or (3′) is merely a technicality for the purposes of this thesis. It can be shown that a map satisfying (1), (2), and (3′) induces a topology on $K$ via a neighborhood system for 0 consisting of the sets $\{\alpha \in K \mid |\alpha| < \varepsilon\}$ where $\varepsilon \in \mathbb{R}_{>0}$. Moreover, one can prove that every such topology is equivalent to one in which $c = 2$, and that (3′) with $c = 2$ implies the triangle inequality. It is for this reason that we often ignore the technicality posed by (3′) and simply use the triangle inequality instead. This causes no issues since results are generally stated with regards to the places of a number field, rather than specific absolute values. For more details about (3′) and it's relation to the triangle inequality see [1, Ch 2.1] and [6, Ch 1 §1].

We close the section with two standard results from algebraic number theory regarding absolute values on a field and places of an algebraic number field.

**Theorem 1.4** (Weak approximation theorem)**.** *Let $K$ be a field, let $|\cdot|_1, |\cdot|_2, \dots, |\cdot|_n$ be pairwise inequivalent absolute values on $K$, and let $x_1, x_2, \dots, x_n \in K$. Then for every $\varepsilon > 0$ there exists $\lambda \in K$ such that*

$$|\lambda - x_i|_i < \varepsilon,$$

*for all $i \in \{1, \dots, n\}$.*

*Proof.* See Proposition 3.4 in [14, Ch 2 §3]. □

Now, define the signature function $\mathrm{sgn} : \mathbb{R}^\times \to \{\pm 1\}$ as follows

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases}$$

The approximation theorem can be used to prove the following corollary.

**Corollary 1.5.** *Let $K$ be a number field, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be distinct prime ideals of $K$, and let $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$. Furthermore, let $\tau_1, \dots, \tau_m$ be real embeddings of $K$, and let $\varepsilon_1, \dots, \varepsilon_m \in \{\pm 1\}$. Then, there exists $\lambda \in K^\times$ such that*

$$\mathrm{sgn}(\tau_j(\lambda)) = \varepsilon_j,$$

*for all $j \in \{1, \dots, m\}$, and*

$$\mathrm{ord}_{\mathfrak{p}_i}(\lambda) = a_i,$$

*for all $i \in \{1, \dots, n\}$.*

*Proof.* Let $x_i \in \mathfrak{p}_i^{a_i} \setminus \mathfrak{p}_i^{a_i+1}$ for $i \in \{1, \dots, n\}$. Each of these $x_i$ is guaranteed to exist since if $\mathfrak{p}_i^{a_i} = \mathfrak{p}_i^{a_i+1}$ this would imply that $\mathfrak{p}_i = (1)$, a contradiction since each $\mathfrak{p}_i$ is prime. Choosing the $x_i$ in this way immediately implies

$$\mathrm{ord}_{\mathfrak{p}_i}(x_i) = a_i \tag{1.3}$$

for all $i \in \{1, \ldots, n\}$. For each $i$, let $|\cdot|_{\mathfrak{p}_i}$ be the absolute value associated to the prime $\mathfrak{p}_i$ and for each $j$ let $|\cdot|_{\tau_j}$ be the absolute value associated to the real embedding $\tau_j$. Then Theorem 1.4 implies that for all $\varepsilon > 0$ there exists $\lambda \in K$ such that

$$|\lambda - \varepsilon_j|_{\tau_j} < \varepsilon \tag{1.4}$$

for all $j \in \{1, \ldots, m\}$, and also

$$|\lambda - x_i|_{\mathfrak{p}_i} < \varepsilon$$

for all $i \in \{1, \ldots, n\}$. Now, let $\varepsilon < 1$ and also small enough that

$$\operatorname{ord}_{\mathfrak{p}_i}(\lambda - x_i) \geq a_i + 1 \tag{1.5}$$

holds for all $i \in \{1, \ldots, n\}$. First note that since $\varepsilon < 1$ the inequalities in (1.4) imply $\operatorname{sgn}(\tau_j(\lambda)) = \varepsilon_j$ for all $j \in \{1, \ldots, m\}$. Indeed, if $\varepsilon_j = 1$ then

$$|\lambda - \varepsilon_j|_{\tau_j} = |\tau_j(\lambda) - 1| < \varepsilon < 1.$$

Thus, $\tau_j(\lambda) > 0$ so $\operatorname{sgn}(\tau_j(\lambda)) = 1 = \varepsilon_j$. An analogous argument applies when $\varepsilon_j = -1$. This proves the first group of identities given in the statement of the corollary.

For the second, recall that our choice of epsilon ensures (1.5) holds. This implies that $\operatorname{ord}_{\mathfrak{p}_i}(\lambda) = \operatorname{ord}_{\mathfrak{p}_i}(x_i) = a_i$ as desired. Indeed, if $\operatorname{ord}_{\mathfrak{p}_i}(\lambda) \neq \operatorname{ord}_{\mathfrak{p}_i}(x_i)$ then we would have

$$\operatorname{ord}_{\mathfrak{p}_i}(\lambda - x_i) = \min\{\operatorname{ord}_{\mathfrak{p}_i}(\lambda), \operatorname{ord}_{\mathfrak{p}_i}(x_i)\} \leq \operatorname{ord}_{\mathfrak{p}_i}(x_i) = a_i,$$

where the last equality follows from (1.3). But this contradicts (1.5), so in fact $\operatorname{ord}_{\mathfrak{p}_i}(\lambda) = a_i$ for all $i \in \{1, \ldots, n\}$. □

## 1.4 Characters modulo $\mathfrak{m}$ and Gauss Sums

Given a group $G$, a character on $G$ is defined as a group morphism $\chi : G \to \mathbb{C}^\times$. The notation $\widehat{G}$ is used to denote the group of characters of $G$ with pointwise multiplication. Given a character $\chi \in G$ we define the complex conjugate character

$\overline{\chi} : G \to \mathbb{C}^\times$ by $\overline{\chi}(g) = \overline{\chi(g)}$ for all $g \in G$. An important fact that we will commonly make use of is the following. If $G$ is a finite group with $n = |G|$, and $g \in G$ then

$$\chi(g)^n = \chi(g^n) = 1.$$

Thus, when $G$ is finite $\chi(g)$ is necessarily a root of unity. This means that

$$\overline{\chi}(g) = \chi(g)^{-1} = \chi(g^{-1})$$

for all $g \in G$.

Now, let $K$ be a number field with ring of integers $\mathcal{O}$, and let $\mathfrak{m}$ be an integral ideal of $K$. A character $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$ is called a character modulo $\mathfrak{m}$. Note that the natural projection $\pi : \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ sends the elements $\alpha$ of $\mathcal{O}$ with $(\alpha, \mathfrak{m}) = 1$ to $(\mathcal{O}/\mathfrak{m})^\times$. Thus, given $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$ we may extend it to a character on the set of algebraic integers prime to $\mathfrak{m}$ by composing it with $\pi$. From here, we may extend the character to a multiplicative function on all algebraic integers $\alpha$ by sending any $\alpha$ with $(\alpha, \mathfrak{m}) \neq 1$ to zero. For notational simplicity, both of these related characters as well as the multiplicative function are all denoted by $\chi$.

Given an integral ideal $\mathfrak{g}$ dividing $\mathfrak{m}$ there is a well defined group morphism $\pi_{\mathfrak{g}} : \mathcal{O}/\mathfrak{m} \to \mathcal{O}/\mathfrak{g}$, obtained by projection. This morphism sends $(\mathcal{O}/\mathfrak{m})^\times$ to $(\mathcal{O}/\mathfrak{g})^\times$. Thus, a character $\psi \in \widehat{(\mathcal{O}/\mathfrak{g})^\times}$ can always be extended to a character on $(\mathcal{O}/\mathfrak{m})^\times$ by composing with $\pi_{\mathfrak{g}}$. A character $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$ is called a proper character (or a primitive character) if it does not arise in this way from a character on $(\mathcal{O}/\mathfrak{g})^\times$ for any $\mathfrak{g}$ properly dividing $\mathfrak{m}$.

**Lemma 1.6.** *A character $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$ is proper if and only if for each integral ideal $\mathfrak{g}$ properly dividing $\mathfrak{m}$ there exist $\alpha, \beta \in \mathcal{O}$ both prime to $\mathfrak{m}$ and such that $\alpha \equiv \beta \pmod{\mathfrak{g}}$ but $\chi(\alpha) \neq \chi(\beta)$.*

*Proof.* This fact is immediate by considering an arbitrary integral ideal $\mathfrak{g}$ properly dividing $\mathfrak{m}$ and then examining the diagram below.

$$\begin{array}{ccc} (\mathcal{O}/\mathfrak{m})^\times & \xrightarrow{\chi} & \mathbb{C}^\times \\ {\scriptstyle \pi_{\mathfrak{g}}}\downarrow & \nearrow_{\psi} & \\ (\mathcal{O}/\mathfrak{g})^\times & & \end{array}$$

If $\chi$ is proper then by definition there does not exist any character $\psi$ that makes the diagram commute, so there must exist $\alpha, \beta \in \mathcal{O}$ with the properties outlined in the lemma. Conversely, if there exist $\alpha, \beta \in \mathcal{O}$ with the properties outlined in the lemma, then again there does not exist any $\psi$ that makes the diagram commute, so $\chi$ is proper. □

We now introduce the Gauss sum associated to a character modulo $\mathfrak{m}$ and prove a few important results about it that will be useful later. Recall that the different ideal $\mathfrak{d} := \mathfrak{d}_{K/\mathbb{Q}}$ is defined via

$$\mathfrak{d}^{-1} := \{\lambda \in K \mid \mathrm{Tr}(\lambda\mathcal{O}) \subseteq \mathbb{Z}\},$$

where $\mathrm{Tr} := \mathrm{Tr}_{K/\mathbb{Q}}$ is the absolute trace. It is a standard fact from algebraic number theory that $\mathfrak{d}$ is an integral ideal.

**Definition 1.7.** Let $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})}^\times$ and let $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$. Define the Gauss sum

$$\begin{aligned}\tau(\chi, y) &= \sum_{z \in (\mathcal{O}/\mathfrak{m})^\times} \chi(z) e^{2\pi i Tr(zy)} \\ &= \sum_{z \in \mathcal{O}/\mathfrak{m}} \chi(z) e^{2\pi i Tr(zy)}.\end{aligned}$$

Recall that $\chi(z)$ is defined to be zero for all $z \in \mathcal{O}$ with $(z, \mathfrak{m}) \neq 1$, which justifies the second equality above.

The Gauss sum $\tau(\chi, y)$ is well defined because it's value is independent of the choice of representatives $z$. Indeed if $\tilde{z} \equiv z \pmod{\mathfrak{m}}$ then $\tilde{z} - z \in \mathfrak{m}$ so

$$\tilde{z}y - zy \in \mathfrak{m}\mathfrak{m}^{-1}\mathfrak{d}^{-1} = \mathfrak{d}^{-1}.$$

Thus, $\mathrm{Tr}(\tilde{z}y - zy) \in \mathbb{Z}$ which implies

$$e^{2\pi i Tr(\tilde{z}y)} = e^{2\pi i Tr(zy)}.$$

Also note that if $\alpha \in \mathcal{O}$ and $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ then $\alpha y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$. So, $\tau(\chi, \alpha y)$ is also well defined. We now prove a useful lemma that will help establish some important results about Gauss sums.

**Lemma 1.8.** *Let $\alpha \in \mathcal{O}$ and let $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ be such that $(y\mathfrak{m}\mathfrak{d}, \mathfrak{m}) = 1$. Then*

$$\sum_{z \in \mathcal{O}/\mathfrak{m}} e^{2\pi i\, Tr(z\alpha y)} = \begin{cases} 0, & \text{if } \alpha \not\equiv 0 \pmod{\mathfrak{m}}; \\ \mathbb{N}(\mathfrak{m}), & \text{if } \alpha \equiv 0 \pmod{\mathfrak{m}}. \end{cases}$$

*Proof.* First assume $\alpha \equiv 0 \pmod{\mathfrak{m}}$. Since $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ this means $y\mathfrak{m}\mathfrak{d} = \mathfrak{a}$ for some integral ideal $\mathfrak{a}$. Thus, $y\mathfrak{m} = \mathfrak{a}\mathfrak{d}^{-1} \subseteq \mathfrak{d}^{-1}$, so $y\alpha \in \mathfrak{d}^{-1}$. Thus, since $z \in \mathcal{O}$ we have $\mathrm{Tr}(z\alpha y) \in \mathbb{Z}$. This proves the claim when $\alpha \equiv 0 \pmod{\mathfrak{m}}$.

Now assume $\alpha \not\equiv 0 \pmod{\mathfrak{m}}$. This implies that $y\alpha \notin \mathfrak{d}^{-1}$ as follows. Let $\mathfrak{a} := y\mathfrak{m}\mathfrak{d}$. If $y\alpha \in \mathfrak{d}^{-1}$ then

$$\alpha\mathfrak{a} = \alpha y\mathfrak{m}\mathfrak{d} \subseteq \mathfrak{m}$$

Since $\alpha \in \mathcal{O}$ and $\mathfrak{a}$ is integral this means $\mathfrak{m} \mid \alpha\mathfrak{a}$. But $(\mathfrak{a}, \mathfrak{m}) = 1$ by assumption, so in fact $\mathfrak{m} \mid \alpha$ so $\alpha \equiv 0 \pmod{\mathfrak{m}}$. This is a contradiction, so $y\alpha \notin \mathfrak{d}^{-1}$. We will show that this fact itself leads to a contradiction unless the summation in question is equal to zero. Begin by noting that the map $z \mapsto z - 1$ permutes the elements of $\mathcal{O}/\mathfrak{m}$ and permuting the elements in this way leaves the sum of interest unchanged. That is

$$\begin{aligned} \sum_{z \in \mathcal{O}/\mathfrak{m}} e^{2\pi i \mathrm{Tr}(z\alpha y)} &= \sum_{z \in \mathcal{O}/\mathfrak{m}} e^{2\pi i \mathrm{Tr}((z-1)\alpha y)} \\ &= e^{-2\pi i \mathrm{Tr}(\alpha y)} \sum_{z \in \mathcal{O}/\mathfrak{m}} e^{2\pi i \mathrm{Tr}(z\alpha y)}. \end{aligned}$$

Unless the summation is equal to zero, this means $e^{-2\pi i \mathrm{Tr}(\alpha y)} = 1$, which in turn means $\mathrm{Tr}(\alpha y) \in \mathbb{Z}$. Note that given an arbitrary $\lambda \in \mathcal{O}$ we may replace $y$ by $y\lambda$ in the above argument, showing that in fact $\mathrm{Tr}(\alpha y\lambda) \in \mathbb{Z}$ for all $\lambda \in \mathcal{O}$. Thus, $\alpha y \in \mathfrak{d}^{-1}$. This is a contradiction, which proves the claim when $\alpha \not\equiv 0 \pmod{\mathfrak{m}}$. $\square$

We will now prove the two most important results we will need about Gauss sums.

**Lemma 1.9.** *Let $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$. If $\alpha \in \mathcal{O}$ is prime to $\mathfrak{m}$ then*

$$\tau(\chi, \alpha y) = \overline{\chi}(\alpha)\tau(\chi, y).$$

*Proof.* By definition of the Gauss sum

$$\tau(\chi, \alpha y) = \sum_{z \in \mathcal{O}/\mathfrak{m}} \chi(z) e^{2\pi i \mathrm{Tr}(z\alpha y)}.$$

Because $(\alpha, \mathfrak{m}) = 1$ it follows that $\alpha + \mathfrak{m} \in (\mathcal{O}/\mathfrak{m})^\times$. In particular, it is invertible. Thus, the map $z \mapsto z\alpha$ permutes the elements of $\mathcal{O}/\mathfrak{m}$ and we may write

$$\begin{aligned}\tau(\chi, \alpha y) &= \overline{\chi}(\alpha) \sum_{z \in \mathcal{O}/\mathfrak{m}} \chi(z) e^{2\pi i \mathrm{Tr}(zy)} \\ &= \overline{\chi}(\alpha)\tau(\chi, y).\end{aligned}$$

□

**Lemma 1.10.** *Let $\chi \in \widehat{(\mathcal{O}/\mathfrak{m})^\times}$ be proper, and let $\alpha \in \mathcal{O}$.*

*(1) If $(\alpha, \mathfrak{m}) \neq 1$, then $\tau(\chi, \alpha y) = 0$.*

*(2) If $(\alpha, \mathfrak{m}) = 1$, then $|\tau(\chi, \alpha y)| = \sqrt{\mathbb{N}(\mathfrak{m})}$.*

*Proof.* First suppose $(\alpha, \mathfrak{m}) = \mathfrak{g}$ where $\mathfrak{g} \neq 1$. Write

$$(\alpha) = \mathfrak{a}\mathfrak{g} \quad \text{and} \quad \mathfrak{m} = \mathfrak{b}\mathfrak{g}. \tag{1.6}$$

By assumption $\mathfrak{b}$ properly divides $\mathfrak{m}$. Since $\chi$ is proper, Lemma 1.6 says that there exist $\lambda, \mu \in \mathcal{O}$ both relatively prime to $\mathfrak{m}$ such that $\lambda \equiv \mu \pmod{\mathfrak{b}}$ and such that $\chi(\lambda) \neq \chi(\mu)$. However, applying Lemma 1.9 gives

$$\tau(\chi, \alpha\lambda y) = \overline{\chi}(\lambda)\tau(\chi, \alpha y) \quad \text{and} \quad \tau(\chi, \alpha\mu y) = \overline{\chi}(\mu)\tau(\chi, \alpha y).$$

Since $\lambda - \mu \in \mathfrak{b}$ and (1.6) gives $(\alpha)\mathfrak{b} = \mathfrak{m}\mathfrak{a}$ this means $\alpha(\lambda - \mu) \in \mathfrak{m}$. Thus,

$$\mathrm{Tr}(\alpha(\lambda - \mu)y) \in \mathbb{Z}$$

by using the same argument in the beginning of the proof of Lemma 1.8. This means $\tau(\chi, \alpha\lambda y) = \tau(\chi, \alpha\mu y)$. Thus, unless $\tau(\chi, \alpha y) = 0$, we have that $\overline{\chi}(\lambda) = \overline{\chi}(\mu)$, which is a contradiction, proving the first statement of the lemma.

Now, assume $(\alpha, \mathfrak{m}) = 1$. In this case, using Lemma 1.9 gives

$$|\tau(\chi, \alpha y)|^2 = |\overline{\chi}(\alpha)\tau(\chi, y)|^2 = |\tau(\chi, y)|^2, \tag{1.7}$$

a result which will be useful shortly. Given an arbitrary $z \in \mathcal{O}/\mathfrak{m}$, and using the definition of a Gauss sum it follows that

$$\tau(\chi, zy)\overline{\tau(\chi, zy)} = \sum_{v,w \in (\mathcal{O}/\mathfrak{m})^\times} \chi(v)\overline{\chi}(w) e^{2\pi i \mathrm{Tr}((v-w)zy)}. \tag{1.8}$$

We will now sum both sides of (1.8) over all $z \in \mathcal{O}/\mathfrak{m}$ and compare the left and right hand sides to achieve the desired result. First, consider the left hand side. From the first statement of this lemma, $\tau(\chi, zy)\overline{\tau(\chi, zy)} = 0$ if $(z, \mathfrak{m}) \neq 1$, so summing over $z \in \mathcal{O}/\mathfrak{m}$ produces

$$\begin{aligned} \sum_{z \in \mathcal{O}/\mathfrak{m}} \tau(\chi, zy)\overline{\tau(\chi, zy)} &= \sum_{z \in (\mathcal{O}/\mathfrak{m})^\times} |\tau(\chi, zy)|^2 \\ &= \varphi(\mathfrak{m})|\tau(\chi, y)|^2, \end{aligned} \tag{1.9}$$

where $\varphi(\mathfrak{m}) = |(\mathcal{O}/\mathfrak{m})^\times|$ and the last line follows from (1.7).

We now turn attention toward the right hand side of (1.8), summing it over all elements of $z \in \mathcal{O}/\mathfrak{m}$ to produce

$$\sum_{v,w \in (\mathcal{O}/\mathfrak{m})^\times} \chi(v)\overline{\chi}(w) \sum_{z \in \mathcal{O}/\mathfrak{m}} e^{2\pi i \mathrm{Tr}((v-w)zy)}.$$

Using Lemma 1.8 with $\alpha = v - w$ it follows that when $v \equiv w \pmod{\mathfrak{m}}$ the sum over $z \in \mathcal{O}/\mathfrak{m}$ is equal to $\mathbb{N}(\mathfrak{m})$. When $v \not\equiv w \pmod{\mathfrak{m}}$ then the sum is equal to zero. Thus, the right hand side of (1.8) is

$$\begin{aligned} \sum_{v \in (\mathcal{O}/\mathfrak{m})^\times} \chi(v)\overline{\chi}(v)\mathbb{N}(\mathfrak{m}) &= \sum_{v \in (\mathcal{O}/\mathfrak{m})^\times} \mathbb{N}(\mathfrak{m}) \\ &= \varphi(\mathfrak{m})\mathbb{N}(\mathfrak{m}). \end{aligned}$$

Using this along with (1.9) proves the second statement of the lemma. □

## 1.5 Results from Class Field Theory

The purpose of this section is to state some of the main results of class field theory, which later arguments will rely on. All of these results are stated without proof. For proofs see any standard exposition on class field theory such as in [1], [8], [11], or [14].

We begin by making some definitions that will be used throughout the section. Let $k$ be a number field with ring of integers $\mathcal{O}_k$, and let $\mathfrak{m}$ an integral ideal of $k$. Given $\alpha \in k$, we use the notation

$$\alpha \equiv 1 \pmod{^\times \mathfrak{m}}$$

to mean that $\mathrm{ord}_\mathfrak{p}(\alpha - 1) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{m})$ for all prime ideals $\mathfrak{p} \mid \mathfrak{m}$. Note that if $\alpha \in \mathcal{O}_k$ then this is equivalent to stating $\alpha \equiv 1 \pmod{\mathfrak{m}}$. We say that an element $\lambda \in k^\times$ is totally positive if $\sigma(\lambda) > 0$ for all real embeddings $\sigma$ of $k$.

Let $I_\mathfrak{m}(k)$ denote the group of fractional ideals of $k$ relatively prime to $\mathfrak{m}$, and let $R_\mathfrak{m}^{res}(k)$ denote the ray modulo $\mathfrak{m}$ in the restricted (or narrow) sense. The ray consists of all $\lambda \in k^\times$ with the following properties:

(1) $\lambda \equiv 1 \pmod{^\times \mathfrak{m}}$

(2) $\lambda$ is totally positive.

The set of principal fractional ideals of $k$ generated by elements of the ray $R_\mathfrak{m}^{res}(k)$ is denoted $S_\mathfrak{m}^{res}(k)$. It can be verified that the set $S_\mathfrak{m}^{res}(k)$ is a subgroup of the abelian group $I_\mathfrak{m}(k)$ and so we may define its quotient $Cl_\mathfrak{m}^{res}(k) = I_\mathfrak{m}(k)/S_\mathfrak{m}^{res}(k)$. This is a finite group, and it is referred to as the restricted ray class group modulo $\mathfrak{m}$, or the narrow ray class group modulo $\mathfrak{m}$.

If requirement (1) above is removed, there is a construction analogous to that of $Cl_\mathfrak{m}^{res}(k)$. We write $I(k)$ for the group of fractional ideals of $k$, and $S^{res}(k)$ for the group of principal fractional ideals generated by totally positive elements of $k$. The corresponding class group is $Cl^{res}(k) = I(k)/S^{res}(k)$.

If instead requirement (2) is removed, then this leads to the notion of the ray class group in the unrestricted sense. We write $R_\mathfrak{m}(k)$ for the set of $\lambda \in k^\times$ satisfying property (1). Correspondingly, $S_\mathfrak{m}(k)$ denotes the subgroup of principal ideals generated by elements in $R_{\mathfrak{m}}(k)$, and $Cl_{\mathfrak{m}}(k) = I_{\mathfrak{m}}(k)/S_{\mathfrak{m}}(k)$ is called the ray class group modulo $\mathfrak{m}$.

The groups $Cl^{res}_{\mathfrak{m}}(k)$ and $Cl^{res}(k)$ fit into a useful exact sequence to be introduced in Theorem 1.11. We now define the maps $\rho$ and $\phi$ of this sequence. Let $k^+_{\mathfrak{m}}$ denote the elements of $k$ which are totally positive and relatively prime with $\mathfrak{m}$. One can show that given $x \in k^+_{\mathfrak{m}}$ there exist $\alpha, \beta \in \mathcal{O}_k$ such that $\alpha$ and $\beta$ are each relatively prime with $\mathfrak{m}$ and $x = \alpha/\beta$. Define a map $\rho : k^+_{\mathfrak{m}} \to (\mathcal{O}_k/\mathfrak{m})^\times$ by setting

$$\rho(x) = (\alpha + \mathfrak{m})(\beta + \mathfrak{m})^{-1}.$$

It can be shown that this is a well defined, surjective group morphism. Thus, given any element $\alpha + \mathfrak{m} \in (\mathcal{O}_k/\mathfrak{m})^\times$ there exists an element $x_\alpha \in k^+_{\mathfrak{m}}$ such that $\rho(x_\alpha) = \alpha + \mathfrak{m}$. Define $\phi : (\mathcal{O}_k/\mathfrak{m})^\times \to Cl^{res}_{\mathfrak{m}}(k)$ via the rule

$$\phi(\alpha + \mathfrak{m}) = [x_\alpha \mathcal{O}_k],$$

where $[x_\alpha \mathcal{O}_k]$ is the class of $x_\alpha \mathcal{O}_k$ in $Cl^{res}_{\mathfrak{m}}(k)$.

**Theorem 1.11.** *Let $E^{res}(k)$ denote the group of units in $k$ that are totally positive and let $E^{res}_{\mathfrak{m}}(k)$ denote the group of units in $k$ which are totally positive and congruent to 1 modulo $\mathfrak{m}$. Then there is an exact sequence*

$$1 \longrightarrow E^{res}_{\mathfrak{m}}(k) \longrightarrow E^{res}(k) \xrightarrow{\rho} (\mathcal{O}_k/\mathfrak{m})^\times \xrightarrow{\phi} Cl^{res}_{\mathfrak{m}}(k) \longrightarrow Cl^{res}(k) \longrightarrow 1$$

*where $\rho$ is the restriction of the map defined above to $E^{res}(k)$.*

We briefly remark that in the case when $k$ is quadratic imaginary $\phi$ can be described more simply as

$$\phi(\alpha + \mathfrak{m}) = (\alpha)S_{\mathfrak{m}}(k). \tag{1.10}$$

Indeed, if $k$ is quadratic imaginary all elements are totally positive. Thus, $\alpha \in k^+_{\mathfrak{m}}$ and so, using the notation introduced when defining $\rho$, we may let $x_\alpha = \alpha$. From this, (1.10) follows immediately from the definition of $\rho$.

We may now move on to the main topic of this section: the Artin map. We will define this map and state its important properties. Let $K/k$ be an abelian extension of number fields. For the remainder of this section assume that $\mathfrak{m}$ is divisible by all finite primes which ramify in $K/k$. There are only finitely many such primes in any extension of number fields, so this definition makes sense. Let $\mathfrak{a} \in I_{\mathfrak{m}}(k)$. Then the prime factorization of $\mathfrak{a}$ is of the form

$$\mathfrak{a} = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

where each $n_{\mathfrak{p}} \in \mathbb{N}$ and $n_{\mathfrak{p}} = 0$ for all but finitely many primes $\mathfrak{p}$.

Recall from Section 1.2 that since $K/k$ is abelian, each prime which is unramified in $K/k$ has a unique Frobenius automorphism $\sigma_{\mathfrak{p}}$. Define the Artin symbol

$$\left(\frac{K/k}{\mathfrak{a}}\right) := \prod_{\mathfrak{p} \nmid \mathfrak{m}} \sigma_{\mathfrak{p}}^{n_{\mathfrak{p}}}.$$

The Artin symbol is often written as $(K/k, \mathfrak{a})$ or $\sigma_{\mathfrak{a}}$ when the extension of number fields is understood. Finally, define the Artin map by

$$\begin{aligned} \varphi_{\mathfrak{m}} : I_{\mathfrak{m}}(k) &\to \mathrm{Gal}(K/k) \\ \mathfrak{a} &\mapsto \left(\frac{K/k}{\mathfrak{a}}\right). \end{aligned}$$

**Theorem 1.12.**

*(1) Let $k$ be a number field and $\mathfrak{m}$ an integral ideal of $k$. Then there exists a number field $k_{\mathfrak{m}}^{res}$ called the ray class field modulo $\mathfrak{m}$ (in the restricted sense) for which the Artin map induces an isomorphism*

$$\varphi_{\mathfrak{m}} : Cl_{\mathfrak{m}}^{res}(k) \xrightarrow{\sim} Gal(k_{\mathfrak{m}}^{res}/k). \tag{1.11}$$

*(2) Let $K/k$ be an abelian extension of number fields. There exists an integral ideal $\mathfrak{m}$ of $k$ for which $k_{\mathfrak{m}}^{res} \supseteq K$.*

**Definition 1.13.** Let $K/k$ be an abelian extension of number fields. Define the conductor $\mathfrak{f}$ of the extension to be the smallest (in the sense of divisibility) integral ideal $\mathfrak{m}$ of $k$ satisfying $K \subseteq k_{\mathfrak{m}}^{res}$.

**Theorem 1.14.** *Let $K/k$ be a finite abelian extension with conductor $\mathfrak{f}$. Then a finite prime $\mathfrak{p}$ of $k$ ramifies in $K/k$ if and only if $\mathfrak{p} \mid \mathfrak{f}$.*

## 1.6 Characters of $Cl_{\mathfrak{m}}^{res}(k)$

Let $k$ be a number field with ring of integers $\mathcal{O}$, and let $\mathfrak{m}$ be an integral ideal of $k$. In this section we describe two characters, denoted $\psi_f$ and $\psi_\infty$, which arise from a character $\psi : Cl_{\mathfrak{m}}^{res}(k) \to \mathbb{C}^\times$. The exposition here specializes that of [14, Ch 7 §6], which treats the more general setting where $\psi$ is a Grössencharakter (also called a Hecke character) on $I_{\mathfrak{m}}(k)$.

### 1.6.1 The Character $\psi_f$ and Proper Characters on $Cl_{\mathfrak{m}}^{res}(k)$

Given a character $\psi$ of the ray class group $Cl_{\mathfrak{m}}^{res}(k)$ we may extend this to a character $\psi_f : (\mathcal{O}/\mathfrak{m})^\times \to \mathbb{C}^\times$ by composing $\psi$ with the map

$$\phi : (\mathcal{O}/\mathfrak{m})^\times \to Cl_{\mathfrak{m}}^{res}(k)$$

from the exact sequence of Theorem 1.11. We say that a character $\psi$ of $Cl_{\mathfrak{m}}^{res}(k)$ is proper (or primitive) if the corresponding character $\psi_f$ is proper as described in Section 1.4. Note that given an integral ideal $\mathfrak{g} \mid \mathfrak{m}$ there is a well defined group morphism $\pi : Cl_{\mathfrak{m}}^{res}(k) \to Cl_{\mathfrak{g}}^{res}(k)$ obtained via projection. This is analogous to the projection map $\pi_{\mathfrak{g}}$ introduced in Section 1.4. The following theorem provides a criteria on the side of ray class groups that can be used to determine if a character on $Cl_{\mathfrak{m}}^{res}(k)$ is proper.

**Theorem 1.15.** *A character $\psi : Cl_{\mathfrak{m}}^{res}(k) \to \mathbb{C}^\times$ is proper if and only if there does not exist an integral ideal $\mathfrak{g}$ properly dividing $\mathfrak{m}$, and a character $\alpha$ for which the diagram*

$$\begin{array}{ccc} Cl_{\mathfrak{m}}^{res}(k) & \xrightarrow{\psi} & \mathbb{C}^\times \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\alpha} & \\ Cl_{\mathfrak{g}}^{res}(k) & & \end{array}$$

*commutes.*

*Proof.* See [14, Ch 7 §6] Proposition 6.2 □

**Theorem 1.16.** *Let $K/k$ be an abelian extension of number fields with Galois group $G$ and conductor $\mathfrak{f}$. Let $\chi \in \widehat{G}$ and let $\psi$ be the character of $Cl_{\mathfrak{f}}^{res}(k)$ defined by the* *composition*

$$Cl_{\mathfrak{f}}^{res}(k) \xrightarrow{\alpha} G \xrightarrow{\chi} \mathbb{C}^{\times},$$

*where $\alpha$ is the composition of the Artin isomorphism (1.11) and the restriction map sending $G_{\mathfrak{f}}$ onto $G$. If $\chi$ is an injective character then $\psi$ is a proper character.*

*Proof.* Assume for contradiction that $\psi$ is not proper. By Theorem 1.16 this means there exists an integral ideal $\mathfrak{g}$ properly dividing $\mathfrak{f}$ and a character $\widetilde{\psi}$ for which the diagram

$$\begin{array}{ccccc} Cl_{\mathfrak{f}}^{res}(k) & \xrightarrow{\alpha} & G & \xhookrightarrow{\chi} & \mathbb{C}^{\times} \\ \pi\downarrow & & & \nearrow_{\widetilde{\psi}} & \\ Cl_{\mathfrak{g}}^{res}(k) & & & & \end{array} \tag{1.12}$$

commutes. Let $H = \mathrm{Gal}((K \cap k_{\mathfrak{g}})/k)$. Note that if $\widetilde{\psi}$ factors through $H$, then this would produce the diagram

$$\begin{array}{ccccc} Cl_{\mathfrak{f}}^{res}(k) & \xrightarrow{\alpha} & G & \xhookrightarrow{\chi} & \mathbb{C}^{\times} \\ \pi\downarrow & & \downarrow\!\!\downarrow \pi_h & \nearrow_{\widetilde{\psi}} & \\ Cl_{\mathfrak{g}}^{res}(k) & \longrightarrow & H & & \end{array}$$

where we have renamed $\widetilde{\psi}$ to be the induced map on $H$. But if such a $\widetilde{\psi}$ exists then $\ker \pi_h \subseteq \ker \chi$. This means $G \cong H$. So in fact $K = K \cap k_{\mathfrak{g}}$, meaning $k_{\mathfrak{g}} \supseteq K$. This contradicts the fact that $\mathfrak{f}$ is the conductor of $K/k$, which implies that $\psi$ is proper.

Thus, it suffices to show that $\psi$ must factor through $H$. To see why this is the case, we will consider several diagrams related to (1.12). For the sake of keeping notation easily readable, the names of the maps in each of these diagrams will be left as they are in (1.12). It should be clear how each of these maps are induced from the original maps. First, consider the diagram

$$\begin{array}{ccccc} G_{\mathfrak{f}} & \xrightarrow{\alpha} & G & \xhookrightarrow{\chi} & \mathbb{C}^{\times} \\ \pi\downarrow & & & \nearrow_{\widetilde{\psi}} & \\ G_{\mathfrak{g}} & & & & \end{array}$$

produced by replacing the ray class groups from (1.12) with the Galois groups of the corresponding ray class fields. This is justified by composing the maps $\psi$ and $\widetilde{\psi}$ in (1.12) with the Artin isomorphisms. Furthermore, note that $\mathrm{Gal}(k_{\mathfrak{f}}/Kk_{\mathfrak{g}}) \subseteq \ker \psi$

since $\ker\psi$ consists precisely of the elements of $G_{\mathfrak{f}}$ which fix $K$, and $K \subseteq Kk_{\mathfrak{g}}$. Thus, $\psi$ factors through $\mathrm{Gal}(Kk_{\mathfrak{g}}/k)$ and we obtain the diagram

$$\begin{array}{ccccc} \mathrm{Gal}(Kk_{\mathfrak{g}}/k) & \xrightarrow{\alpha} & G & \xhookrightarrow{\chi} & \mathbb{C}^\times \\ {\scriptstyle\pi}\downarrow & & & \nearrow_{\widetilde{\psi}} & \\ G_{\mathfrak{g}} & & & & \end{array}$$

Using the maps from this diagram, it follows that

$$\mathrm{Gal}(Kk_{\mathfrak{g}}/K) \subseteq \ker(\chi \circ \alpha) = \ker(\widetilde{\psi} \circ \pi).$$

Since $\mathrm{Gal}(Kk_{\mathfrak{g}}/K) \cong \mathrm{Gal}(k_{\mathfrak{g}}/(K \cap k_{\mathfrak{g}}))$ via the restriction map $\pi$, this means

$$\mathrm{Gal}(k_{\mathfrak{g}}/(K \cap k_{\mathfrak{g}})) \subseteq \ker(\widetilde{\psi}).$$

Thus, $\widetilde{\psi}$ factors through $H = \mathrm{Gal}((K \cap k_{\mathfrak{g}})/k)$, as desired. □

### 1.6.2 The Character $\psi_\infty$

As before, let $\psi$ be a character of $Cl_{\mathfrak{m}}^{res}(k)$. The character $\psi_\infty$ associated to $\psi$ is a character defined on the multiplicative Minkowski space. We begin by defining the additive Minkowski space as

$$M_k = \prod_{v \in S_\infty} k_v,$$

where $S_\infty$ denotes the set of infinite places of $k$, and $k_v$ denotes the completion of $k$ with respect to the valuation $v$. The multiplicative Minkowski space is then given by

$$M_k^\times = \prod_{v \in S_\infty} k_v^\times.$$

In what follows, tuples $(\lambda_v)_{v \in S_\infty} \in M_k$ will simply be denoted $(\lambda_v)$. Given a real place $v$, let $\tau_v$ denote the corresponding embedding. Similarly, for complex $v$ let $\{\tau_v, \overline{\tau}_v\}$ denote the corresponding pair of conjugate embeddings. We note that $M_k \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, where $r_1$ is the number of real embeddings of $k$ and $r_2$ is half of the number of complex embeddings. Moreover, $k^\times$ can be identified with a subgroup of $M_k^\times$ via the

Minkowski embedding

$$\begin{aligned} \mu : k^\times &\to M_k^\times \\ \lambda &\mapsto (\tau_v(\lambda)). \end{aligned} \tag{1.13}$$

We also define the following subgroup of $M_k^\times$:

$$M_k^+ := \{(\lambda_v) \in M_k^\times \mid \lambda_v > 0 \text{ for all real } v\}.$$

It will be useful to note that $M_k^\times/M_k^+ \cong \mathbb{F}_2^{r_1}$. Indeed, one can define a map

$$M_k^\times \to \prod_{\substack{v \in S_\infty \\ v \text{ real}}} \{\pm 1\},$$

by sending $(\lambda_v) \mapsto (\mathrm{sgn}(\lambda_v))$. This map is surjective by Corollary 1.5, and its kernel is precisely $M_k^+$.

The goal now will be to show how a character $\psi : Cl_{\mathfrak{m}}^{res}(k) \to \mathbb{C}^\times$ induces a character $\psi_\infty : M_k^\times \to \mathbb{C}^\times$. First note that there is a group morphism

$$R_{\mathfrak{m}}(k)/R_{\mathfrak{m}}^{res}(k) \to Cl_{\mathfrak{m}}^{res}(k),$$

which is induced by sending $x \in R_{\mathfrak{m}}(k)$ to the class of $(x)$ in $Cl_{\mathfrak{m}}^{res}(k)$. Thus, the character $\psi$ induces a character on $R_{\mathfrak{m}}(k)/R_{\mathfrak{m}}^{res}(k)$ via composition with this map. Furthermore, there is an isomorphism

$$\rho : R_{\mathfrak{m}}(k)/R_{\mathfrak{m}}^{res}(k) \to M_k^\times/M_k^+.$$

Indeed, by restricting the Minkowski embedding (1.13) to $R_{\mathfrak{m}}(k)$ we obtain

$$R_{\mathfrak{m}}(k) \xrightarrow{\mu} M_k^\times \xrightarrow{\pi} M_k^\times/M_k^+,$$

where $\pi$ is the standard projection map. The claim is that $\ker(\pi \circ \mu) = R_{\mathfrak{m}}^{res}(k)$ and that $\pi \circ \mu$ is surjective. The statement about the kernel is clear from the definition of $M_k^+$. To show that $\pi \circ \mu$ is surjective it suffices to show that for $\varepsilon_1, \ldots, \varepsilon_{r_1} \in \{\pm 1\}$

there exists an $x \in R_{\mathfrak{m}}(k)$ such that

$$\operatorname{sgn}(\tau_i(x)) = \varepsilon_i$$

for all real embeddings $\tau_i$ of $k$. This is a direct consequence of Corollary 1.5. Thus, we obtain the desired isomorphism $\rho$. Putting these observations together, a character $\psi_\infty$ may be defined on $M_k^\times$ via the following composition of maps

$$M_k^\times \xrightarrow{\pi} M_k^\times/M_k^+ \xrightarrow{\rho^{-1}} R_{\mathfrak{m}}(k)/R_{\mathfrak{m}}^{res}(k) \longrightarrow Cl_{\mathfrak{m}}^{res}(k) \xrightarrow{\psi} \mathbb{C}^\times.$$

It is possible to describe $\psi_\infty$ more explicitly by noting that it is in fact a continuous group morphism from $M_k^\times \to S^1$ where

$$S^1 = \{z \in \mathbb{Z} \mid |z| = 1\}.$$

Since $M_k^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$, we can use facts about the form of continuous group morphisms from $\mathbb{R}^\times \to S^1$ and from $\mathbb{C}^\times \to S^1$ in order to describe an explicit form for $\psi_\infty$. This is what we explain now.

The projection map $\pi$ is continuous with respect to the quotient topology that it induces on $M_k^\times/M_k^+$. Moreover, on $M_k^\times/M_k^+$ the quotient topology is the same as the discrete topology. This is due to the fact that $M_k^+$ is open in $M_k^\times$. In fact, the rest of the spaces in the sequence are discrete as well. Indeed, since $Cl_{\mathfrak{m}}^{res}(k)$ is finite it follows that $\psi(Cl_{\mathfrak{m}}^{res}(k)) \subseteq S^1$, so in fact $\psi_\infty(M_k^\times) \subseteq S^1$. This can be refined even further because $M_k^\times/M_k^+ \cong \mathbb{F}_2^{r_1}$, which means $\psi_\infty(M_k^\times) \subseteq \{\pm 1\}$, a finite space. Thus, each of the maps in the sequence are continuous, implying that $\psi_\infty$ is continuous.

The following lemma classifies all of the continuous maps from $M_k^\times \to S^1$.

**Lemma 1.17.** *Let $\psi : M_k^\times \to S^1$ be a continuous character. Then for each $v \in S_\infty$ there exists a unique $q_v \in \mathbb{R}$ and a unique $p_v \in \mathbb{Z}$ such that $p_v \in \{0, 1\}$ if $v$ is real, for which*

$$\psi((\lambda_v)) = \prod_{v \in S_\infty} \left(\frac{\lambda_v}{|\lambda_v|_v}\right)^{p_v} |\lambda_v|_v^{iq_v},$$

*for all $(\lambda_v) \in M_k^\times$.*

*Proof.* Because $M_k^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$, one can prove this lemma by understanding the continuous group morphisms $\mathbb{R}^\times \to S^1$ and $\mathbb{C}^\times \to S^1$. The details can be found in Proposition 6.7 of [14, Ch 7 §6]. □

The characters on $M_k^\times$ which will be of interest to us are the characters $\psi_\infty$ which arise from ray class characters. The form of the characters $\psi_\infty$ is particularly simple, as explained in the following theorem.

**Theorem 1.18.** *Let $\psi$ be a character of $Cl_{\mathfrak{m}}^{res}(k)$. Then for each real $v \in S_\infty$ there exists a unique $p_v \in \{0, 1\}$ for which the character $\psi_\infty$ on $M_k^\times$ is of the form*

$$\psi_\infty((\lambda_v)) = \prod_{\substack{v \in S_\infty \\ v \text{ real}}} \left(\frac{\lambda_v}{|\lambda_v|_v}\right)^{p_v},$$

*for all $(\lambda_v) \in M_k^\times$.*

*Proof.* Since $\psi_\infty$ is continuous it takes the form provided in Lemma 1.17. By choosing specific $(\lambda_v) \in M_k^\times$ and using the fact that $\psi_\infty(M_k^\times) \subseteq \{\pm 1\}$, it can be shown that in fact $q_v = 0$ for all places $v \in S_\infty$ and that $p_v = 0$ for all complex places $v \in S_\infty$.

Fix a place $w \in S_\infty$ and let $\lambda_0 := (\lambda_v)$ where $\lambda_v = 1$ for all $v \neq w$ and $\lambda_w$ is a positive rational number. Then

$$\psi_\infty(\lambda_0) = \left(\frac{\lambda_w}{|\lambda_w|_w}\right)^{p_w} \cdot |\lambda_w|_w^{iq_w} = |\lambda_w|_w^{iq_w} = \pm 1.$$

Since this holds for any positive rational $\lambda_w$, this implies that $q_w = 0$. Moreover, since $w \in S_\infty$ was arbitrary, it is in fact the case that $q_v = 0$ for all $v \in S_\infty$. Thus

$$\psi_\infty((\lambda_v)) = \prod_{v \in S_\infty} \left(\frac{\lambda_v}{|\lambda_v|_v}\right)^{p_v}$$

for all $(\lambda_v) \in M_k^\times$. It remains to show that $p_v = 0$ for all complex $v \in S_\infty$. Fix a complex place $w \in S_\infty$. Redefine $\lambda_0$ to be the same as above, except now let $\lambda_w \in k_w^\times \setminus \mathbb{R}$. Such a $\lambda_w$ is guaranteed to exist since $w$ is a complex place. Then

$$\psi_\infty(\lambda_0) = \left(\frac{\lambda_w}{|\lambda_w|_w}\right)^{p_w} = \pm 1.$$

This implies that $p_w = 0$ since $\lambda_w \notin \mathbb{R}^\times$. Since $w$ was an arbitrary complex place, this means $p_v = 0$ for all complex $v \in S_\infty$, which concludes the proof. □

## 1.7 S-truncated L-functions

The main statement in Stark's abelian rank 1 conjecture concerns special values of L-functions at $s = 0$. Here, we define the relevant L-functions which will be needed to make sense of the statement of the conjecture, as well as its proof.

Let $K/k$ be an abelian extension of number fields with Galois group $G$, and let $\widehat{G}$ denote the set of characters of $G$. Furthermore, let $\mathfrak{p}$ be a prime ideal of $k$ and let $\chi \in \widehat{G}$. The character $\chi$ is said to be unramified at $\mathfrak{p}$ if $I_\mathfrak{p} \subseteq \ker(\chi)$, where $I_\mathfrak{p}$ is the inertia subgroup associated to $\mathfrak{p}$. If $\chi$ is unramified at $\mathfrak{p}$ it induces a character on $G/I_\mathfrak{p}$, which is the Galois group of $K^{I_\mathfrak{p}}/k$. It is a standard fact from algebraic number theory that $\mathfrak{p}$ is unramified in this extension, and it therefore has a unique Frobenius automorphism given by $\sigma_\mathfrak{p} := (K^{I_\mathfrak{p}}/k, \mathfrak{p})$.

For each prime ideal $\mathfrak{p}$ define the Euler factor for $\mathfrak{p}$ associated to $\chi$ as

$$E_\mathfrak{p}(s,\chi) = \begin{cases} \left(1 - \frac{\chi(\sigma_\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1}, & \text{if } \chi \text{ is unramified at } \mathfrak{p}; \\ 1, & \text{otherwise.} \end{cases} \tag{1.14}$$

Furthermore, let $S_\infty$ be the set of infinite places of $k$, and let $S \supseteq S_\infty$ be a finite set of places of $k$. Define the $S$-truncated L-function associated to $\chi$ as

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p} \notin S} E_\mathfrak{p}(s,\chi), \quad \text{Re}(s) > 1.$$

Note that if $\mathfrak{p}$ is unramified in $K/k$ then $I_\mathfrak{p}$ is trivial, so in this case all $\chi \in \widehat{G}$ are unramified at $\mathfrak{p}$ and $K^{I_\mathfrak{p}} = K$. Thus, when $\mathfrak{p}$ is ramified it follows that $\sigma_\mathfrak{p} = (K/k, \mathfrak{p})$. Letting $S_{ram}$ denote the set of finite places of $k$ which ramify in $K/k$, we note that if $S \supseteq S_\infty \cup S_{ram}$ as will be assumed for Stark's conjecture, then

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p} \notin S} \left(1 - \frac{\chi(\sigma_\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Moreover, in this case the $S$-truncated L-function can also be written in terms of an infinite sum as follows

$$L_{K/k,S}(s,\chi) = \sum_{(\mathfrak{a},S)=1} \frac{\chi(\sigma_{\mathfrak{a}})}{\mathbb{N}(\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1,$$

where the sum is taken over all integral ideals of $k$ that are not in $S$, and $\sigma_{\mathfrak{a}} = (K/k, \mathfrak{a})$ is the Artin symbol.

Finally, in the case where $S = S_\infty$ the resulting L-function $L_{K/k,S_\infty}(s,\chi)$ is called the primitive L-function associated to $\chi$ and it is common to write

$$L_{K/k}(s,\chi) := L_{K/k,S_\infty}(s,\chi).$$

**Theorem 1.19.** *Let $K/k$ be an abelian extension of number fields, and let $S$ be a finite set of places such that $S \supseteq S_\infty$. Then*

$$\mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \begin{cases} |S| - 1, & \chi = \chi_1; \\ |\{v \in S \mid G_v \subseteq \ker(\chi)\}|, & \chi \neq \chi_1, \end{cases}$$

*where $G_v$ is the decomposition group associated to the place $v$.*

*Proof.* See [22, Ch 1 §3] □

Now, let $\mathfrak{m}$ be an integral ideal of $k$, and let $\psi : Cl_{\mathfrak{m}}^{res}(k) \to \mathbb{C}^\times$ be a character of the ray class group modulo $\mathfrak{m}$. Note that the character $\psi$ gives rise to a character of $I_{\mathfrak{m}}(k)$. Indeed, we have the composition of maps

$$I_{\mathfrak{m}}(k) \xrightarrow{\pi} Cl_{\mathfrak{m}}^{res}(k) \xrightarrow{\psi} \mathbb{C}^\times$$

where $\pi$ is the canonical projection map. It is clear that $\psi \circ \pi$ is a character of $I_{\mathfrak{m}}(k)$. We will regularly use $\psi$ to refer to both of these characters. That is, if $\psi$ is a character of $Cl_{\mathfrak{m}}^{res}(k)$ and $\mathfrak{a} \in I_{\mathfrak{m}}(k)$ it is understood that $\psi(\mathfrak{a}) := \psi([\mathfrak{a}])$ where $[\mathfrak{a}]$ is the class of $\mathfrak{a}$ in $Cl_{\mathfrak{m}}^{res}(k)$. Define

$$L_{k,\mathfrak{m}}(s,\psi) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \left(1 - \frac{\psi(\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

This notation makes clear the dependence of the L-function on both $k$ and $\mathfrak{m}$. In practice, the field $k$ should be clear from context, and we will simply write $L_{\mathfrak{m}}(s, \psi)$. As with the $S$-truncated L-function defined prior, the function $L_{k,\mathfrak{m}}(s, \psi)$ also has an infinite sum representation

$$L_{k,\mathfrak{m}}(s, \psi) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \frac{\psi(\mathfrak{a})}{\mathbb{N}(\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1,$$

where the sum is taken over integral ideals $\mathfrak{a}$ of $k$.

The two L-functions defined above are related to each other in a few important ways, which are described in the following theorems. In what follows, let $k$ be a number field and $\mathfrak{m}$ be an integral ideal of $k$. Let $\chi \in \widehat{G}^{res}_{\mathfrak{m}}$, where $G^{res}_{\mathfrak{m}} := \mathrm{Gal}(k^{res}_{\mathfrak{m}}/k)$.

**Theorem 1.20.** *Let $S_{\mathfrak{m}} = S_\infty \cup \{\mathfrak{p} \mid \mathfrak{m}\}$ and $\psi = \chi \circ \varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}}$ is the Artin isomorphism (1.11). Then*

$$L_{k^{res}_{\mathfrak{m}}/k, S_{\mathfrak{m}}}(s, \chi) = L_{\mathfrak{m}}(s, \psi).$$

*Proof.* First, recall from Theorem 1.14 that a prime ideal $\mathfrak{p}$ ramifies in an extension if and only if it divides the conductor of the extension. The conductor of $k^{res}_{\mathfrak{m}}/k$ divides $\mathfrak{m}$ so if a prime ramifies in $k^{res}_{\mathfrak{m}}/k$ then it divides $\mathfrak{m}$ and thus $S_{ram} \subseteq \{\mathfrak{p} \mid \mathfrak{m}\}$. This implies, $S_{\mathfrak{m}} \supseteq S_\infty \cup S_{ram}$, so

$$\begin{aligned} L_{k^{res}_{\mathfrak{m}}/k, S_{\mathfrak{m}}}(s, \chi) &= \prod_{\mathfrak{p} \nmid \mathfrak{m}} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1} \\ &= \prod_{\mathfrak{p} \nmid \mathfrak{m}} \left(1 - \frac{\psi(\mathfrak{p})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1} \\ &= L_{\mathfrak{m}}(s, \psi). \end{aligned}$$

□

We next examine the relationship between the two types of L-functions when $S = S_\infty$. Consider the tower of fields $k \subseteq F_\chi \subseteq k^{res}_{\mathfrak{m}}$, where $F_\chi$ is the subfield of $k^{res}_{\mathfrak{m}}$ fixed by $\ker \chi$. The character $\chi$ induces a character $\widetilde{\chi}$ on $\mathrm{Gal}(F_\chi/k)$. Letting $\mathfrak{f}_\chi$ denote the conductor of $F_\chi/k$ there is the following composition of maps

$$Cl^{res}_{\mathfrak{f}_\chi}(k) \xrightarrow{\varphi} G_{\mathfrak{f}_\chi} \xrightarrow{\pi} \mathrm{Gal}(F_\chi/k) \xrightarrow{\widetilde{\chi}} \mathbb{C}^\times,$$

where $\varphi$ is the Artin isomorphism and $\pi$ is just the restriction map. Denote the character defined by this composition as $\psi$.

**Theorem 1.21.** *Let $\psi$ be defined as above. Then*

$$L_{k_{\mathfrak{m}}^{res}/k}(s,\chi) = L_{\mathfrak{f}_\chi}(s,\psi).$$

*Proof.* We begin by explaining that $\chi$ is unramified at precisely the prime ideals $\mathfrak{p}$ for which $\mathfrak{p} \nmid \mathfrak{f}_\chi$. Let $K = k_{\mathfrak{m}}^{res}$ and assume $\chi$ is unramified at $\mathfrak{p}$. Then $I_{\mathfrak{p}} \subseteq \ker\chi$, so $K^{\ker\chi} \subseteq K^{I_{\mathfrak{p}}}$. Thus, since $\mathfrak{p}$ is unramified in $K^{I_{\mathfrak{p}}}/k$ it is also unramified in $K^{\ker\chi}/k$. This means that $\mathfrak{p} \nmid \mathfrak{f}_\chi$ since $\mathfrak{p}$ is ramified in an extension if and only if it divides the conductor of the extension, by Theorem 1.14.

Conversely, assume that $\mathfrak{p} \nmid \mathfrak{f}_\chi$. Then $\mathfrak{p}$ is unramified in $K^{\ker\chi}/k$. Since $K^{I_{\mathfrak{p}}}/k$ is the maximal abelian subextension for which $\mathfrak{p}$ is unramified, this means $K^{\ker\chi} \subseteq K^{I_{\mathfrak{p}}}$ so $I_{\mathfrak{p}} \subseteq \ker\chi$, which means $\chi$ is unramified at $\mathfrak{p}$. Thus,

$$L_{k_{\mathfrak{m}}^{res}/k}(s,\chi) = \prod_{\mathfrak{p}\nmid\mathfrak{f}_\chi}\left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right)^{-1},$$

where $\sigma_{\mathfrak{p}} = (K^{I_{\mathfrak{p}}}/k,\mathfrak{p})$. Note that $\sigma_{\mathfrak{p}}$ restricted to $K^{\ker\chi}$ gives the Frobenius automorphism $(K^{\ker\chi}/k,\mathfrak{p})$. We have

$$\chi(\sigma_{\mathfrak{p}}) = \widetilde{\chi}((K^{\ker\chi}/k,\mathfrak{p})) = (\widetilde{\chi}\circ\pi\circ\varphi)(\mathfrak{p}) = \psi(\mathfrak{p}).$$

This along with the expression for $L_{k_{\mathfrak{m}}^{res}/k}(s,\chi)$ above completes the proof. □

## 1.8 The Functional Equation for the Completed L-function

We now describe a functional equation, which will be used to relate $L_{\mathfrak{m}}(1,\overline{\psi})$ with $L'_{\mathfrak{m}}(0,\psi)$ for proper characters $\psi$. The functional equation here is a specific case of the one described in Theorem 8.5 of [14, Ch 7 §8], which is stated more generally for Grössencharakters.

Begin by recalling the standard definition of the gamma function. For $s \in \mathbb{C}$ with

$\mathrm{Re}(s) > 0$, it is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{1.15}$$

This is an analytic function on the right half plane and can be extended to a meromorphic function on $\mathbb{C}$ with simple poles at the non-positive integers.

Let $k$ be a number field, $\mathfrak{m}$ an integral ideal of $k$, and $\psi$ a proper character of $Cl_{\mathfrak{m}}^{res}(k)$. We also define the following modified gamma functions:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Then, for each place $v$ of $k$ recall from Section 1.6 that $\psi$ induces a character $\psi_\infty$ on the Minkowski space $M_k^\times$, and this character is of the form

$$\psi_\infty((\lambda_v)) = \prod_{\substack{v \in S_\infty \\ v \text{ real}}} \left(\frac{\lambda_v}{|\lambda_v|_v}\right)^{p_v},$$

where $p_v \in \{0, 1\}$ is uniquely determined by $v$. Define

$$E_v(s, \psi) = \begin{cases} \Gamma_{\mathbb{C}}(s), & \text{if } v \text{ is complex;} \\ \Gamma_{\mathbb{R}}(s + p_v), & \text{if } v \text{ is real.} \end{cases}$$

Moreover, let

$$L_\infty(s, \psi) = \prod_{v \in S_\infty} E_v(s, \psi).$$

The completed L-function associated to the proper character $\psi$ is defined as

$$\Lambda(s, \psi) = (|d_k|\mathbb{N}(\mathfrak{m}))^{s/2} L_\infty(s, \psi) L_{\mathfrak{m}}(s, \psi).$$

**Theorem 1.22.** *If $\psi$ is a proper character of $Cl_{\mathfrak{m}}^{res}(k)$, then the completed L-function $\Lambda(s, \psi)$ satisfies the functional equation*

$$\Lambda(s, \psi) = W(\psi)\Lambda(1 - s, \overline{\psi}),$$

*for some $W(\psi) \in \mathbb{C}^\times$ satisfying $|W(\psi)| = 1$.*

*Proof.* See Theorem 8.5 and Corollary 8.6 of [14, Ch 7 §8]. □

Ultimately, the case in which $k$ is quadratic imaginary will be the one of interest. More specifically, the functional equation above will be used as an aid in relating $L_{\mathfrak{m}}(1,\overline{\chi})$ to $L'_{\mathfrak{m}}(0,\psi)$. The following corollary of Theorem 1.22 explains how this is done. First, recall that $\tau(\psi_f, y)$ is the Gauss sum from Definition 1.7, where $\psi_f : (\mathcal{O}_k/\mathfrak{m})^\times \to \mathbb{C}^\times$ is the finite character associated to $\psi$ as defined in Section 1.6, and $y$ is any element of $\mathfrak{m}^{-1}\mathfrak{d}^{-1}$ with $(y, y\mathfrak{m}\mathfrak{d}) = 1$.

**Corollary 1.23.** *If $k$ is quadratic imaginary and $\psi$ is a proper character of $Cl_{\mathfrak{m}}^{res}(k)$ then*

$$L'_{\mathfrak{m}}(0,\psi) = W(\psi)\frac{\sqrt{|d_k|\mathbb{N}(\mathfrak{m})}}{2\pi}L_{\mathfrak{m}}(1,\overline{\psi}),$$

*where*

$$W(\psi) = \frac{\psi(y\mathfrak{m}\mathfrak{d})\tau(\psi_f, y)}{\sqrt{\mathbb{N}(\mathfrak{m})}}.$$

*Proof.* The expression for $W(\psi)$ can be deduced from Theorem 8.5 of [14, Ch 7 §8]. Since $k$ is quadratic imaginary there is one unique complex place and no real places. This means that $L_\infty(s,\psi) = \Gamma_{\mathbb{C}}(s)$. Thus, it follows that

$$\Lambda(s,\psi) = 2(2\pi)^{-s}(|d_k|\mathbb{N}(\mathfrak{m}))^{s/2}\Gamma(s)L_{\mathfrak{m}}(s,\psi).$$

Replacing $\Lambda(s,\psi)$ with this expression in the functional equation of Theorem 1.22, and then rearranging for $L_{\mathfrak{m}}(s,\psi)$ gives

$$L_{\mathfrak{m}}(s,\psi) = W(\psi)(2\pi)^{2s-1}(|d_k|\mathbb{N}(\mathfrak{m}))^{(1-2s)/2}\frac{\Gamma(1-s)}{\Gamma(s)}L_{\mathfrak{m}}(1-s,\overline{\psi}). \tag{1.16}$$

The order of vanishing of $L_{\mathfrak{m}}(s,\psi)$ is at least 1 at $s = 0$, a fact which follows directly from Theorem 1.19 and Theorem 1.20. Thus, it follows that

$$L'_{\mathfrak{m}}(0,\psi) = \lim_{s\to 0}\frac{L_{\mathfrak{m}}(s,\psi)}{s}.$$

Dividing both sides of (1.16) by $s$ and then taking the limit as $s \to 0$ immediately produces the desired result because $\Gamma(1) = 1$ and

$$\lim_{s\to 0} s\Gamma(s) = 1$$

since $\Gamma(s)$ has a simple pole of residue 1 at $s = 0$. $\square$

# 2 The Rank One Abelian Stark Conjecture

## 2.1 Statement of the Conjecture

**Conjecture 2.1** (Stark). *Let $K/k$ be an abelian extension of number fields with Galois group $G$ and let $S$ be a finite set of places of $k$ satisfying the following properties:*

*(1)* $S \supseteq S_\infty \cup S_{ram}$

*(2)* $|S| \geq 2$

*(3)* *$S$ contains at least one place $v_0$, which splits completely in $K$.*

*Fix an arbitrary place $w_0$ of $K$ lying above $v_0$, let $w_K$ be the number of roots of unity in $K$, and let $S_K$ be the set of places of $K$ which lie above the places in $S$. Then there exists an $S_K$-unit $\varepsilon$ such that*

$$L'_{K/k,S}(0,\chi) = -\frac{1}{w_K}\sum_{\sigma \in G}\chi(\sigma)\log|\varepsilon^\sigma|_{w_0} \tag{2.1}$$

*for all $\chi \in \widehat{G}$. Moreover,*

- *If $|S| \geq 3$ then $|\varepsilon|_w = 1$ for all places $w$ not lying above $v_0$.*
- *If $|S| = 2$ then $|\varepsilon|_{w^\sigma} = |\varepsilon|_w$ for all places $w$ not lying above $v_0$ and for all $\sigma \in G$.*
- *The extension $K(\varepsilon^{1/w_K})/k$ is abelian.*

We now make a few clarifying remarks regarding terminology and notation in relation to this conjecture.

(1) The statement of Conjecture 2.1 will often be referred to by $St(K/k, S, v_0)$ or by $St(K/k, S, v_0, w_0)$ to emphasize the choice of $w_0$ lying above $v_0$.

(2) An $S_K$-unit $\varepsilon$ as defined in Conjecture 2.1 is called a Stark unit. Stark units are unique up to the roots of unity, as described in Theorem 2.2 below.

(3) Given an $\alpha \in K^\times$, if the extension $K(\alpha^{1/w_K})/k$ is abelian we say that $\alpha$ is $w_K$-abelian over $k$.

**Theorem 2.2.** *If $\varepsilon$ and $\tilde{\varepsilon}$ are Stark units for $St(K/k, S, v_0, w_0)$, then there exists a root of unity $\zeta \in K^\times$ such that $\tilde{\varepsilon} = \zeta\varepsilon$.*

*Proof.* First, we convert (2.1) into another form. Note that

$$\begin{aligned} L_{K/k,S}(s,\chi) &= \sum_{(\mathfrak{a},S)=1} \frac{\chi(\sigma_{\mathfrak{a}})}{\mathbb{N}(\mathfrak{a})^s} \\ &= \sum_{\sigma\in G} \chi(\sigma) \sum_{\substack{(\mathfrak{a},S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \frac{1}{\mathbb{N}(\mathfrak{a})^s} \\ &= \sum_{\sigma\in G} \chi(\sigma)\zeta_{K/k,S}(s,\sigma), \end{aligned}$$

where

$$\zeta_{K/k,S}(s,\sigma) = \sum_{\substack{(\mathfrak{a},S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \frac{1}{\mathbb{N}(\mathfrak{a})^s}$$

is the partial zeta function. Taking the derivative and using this expression for $L'_{K/k,S}(0,\chi)$ in (2.1) gives

$$\sum_{\sigma\in G} \chi(\sigma)\zeta'_{K/k,S}(0,\sigma) = -\frac{1}{w_K} \sum_{\sigma\in G} \chi(\sigma) \log|\varepsilon^\sigma|_{w_0}.$$

Fixing $\tau \in G$, multiplying both sides by $\overline{\chi}(\tau)$, and then summing both sides over all characters in $\widehat{G}$ we have

$$\sum_{\sigma\in G} \zeta'_{K/k,S}(0,\sigma) \sum_{\chi\in\widehat{G}} \chi(\sigma)\overline{\chi}(\tau) = -\frac{1}{w_K} \sum_{\sigma\in G} \log|\varepsilon^\sigma|_{w_0} \sum_{\chi\in\widehat{G}} \chi(\sigma)\overline{\chi}(\tau).$$

It is a standard fact from character theory of finite abelian groups that

$$\sum_{\chi\in\widehat{G}} \chi(\sigma)\overline{\chi}(\tau) = \begin{cases} |G|, & \text{if } \sigma = \tau; \\ 0, & \text{if } \sigma \neq \tau. \end{cases}$$

Since $\tau \in G$ was arbitrary, this shows that (2.1) is equivalent to saying

$$\zeta'_{K/k,S}(0,\sigma) = -\frac{1}{w_K} \log |\varepsilon^{\sigma}|_{w_0},$$

for all $\sigma \in G$. Now, let $\widetilde{\varepsilon}$ be another Stark unit. We claim that

$$\left|\frac{\tilde{\varepsilon}}{\varepsilon}\right|_w = 1 \tag{2.2}$$

for all places $w$ of $K$. Since $\varepsilon$ and $\widetilde{\varepsilon}$ are $S_K$-units, this is immediately true for all places not lying above a place in $S$. For the places lying above $v_0$, consider the expression for $\zeta'_{K/k,S}(0,\sigma)$ above. From this it can be immediately seen that

$$|\varepsilon^{\sigma}|_{w_0} = |\tilde{\varepsilon}^{\sigma}|_{w_0}.$$

Rearranging produces

$$1 = \left|\left(\frac{\tilde{\varepsilon}}{\varepsilon}\right)^{\sigma}\right|_{w_0} = \left|\frac{\tilde{\varepsilon}}{\varepsilon}\right|_{w_0^{\sigma^{-1}}}.$$

Since this holds for all $\sigma \in G$ and $G$ acts transitively on the set of places lying above $v_0$, it follows that (2.2) holds for all places lying above $v_0$.

All that remains is to verify the identity for the places $w \in S_K$ which do not lie above $v_0$. If $|S| \geq 3$ then this is obvious from the absolute value conditions of Stark's conjecture. If $|S| = 2$ then $S$ contains only one place other than $v_0$. Call this place $v$. Then the product formula (1.2) implies

$$1 = \prod_{\substack{w\in S_K \\ w|v}} \left|\frac{\widetilde{\varepsilon}}{\varepsilon}\right|_w.$$

But since $|\varepsilon|_{w^{\sigma}} = \varepsilon_w$ for all $\sigma$ we can again use the fact that $G$ acts transitively on the set of places above $v$ to conclude that

$$1 = \prod_{\sigma \in G} \left| \frac{\widetilde{\varepsilon}}{\varepsilon} \right|_{w^\sigma} = \left| \frac{\widetilde{\varepsilon}}{\varepsilon} \right|_w^r,$$

where $w$ is an arbitrary place lying above $v$ and $r$ is the total number of places above $v$. This establishes (2.2) for all places of $K$.

An $S$-unit which has absolute value 1 for all places in $S$ is necessarily a root of unity. This is a result attributed to Kronecker (see Lemma 1.6 of [23] for a proof). This implies that $\widetilde{\varepsilon} = \zeta\varepsilon$ for some root of unity $\zeta$, as desired. □

Now following Tate [22], we will prove some important lemmas regarding Conjecture 2.1, which will serve to focus later discussion.

**Lemma 2.3.** *The set $\Omega = \{\alpha \in K^\times \mid \alpha$ is $w_K$-abelian$\}$ is a $\mathbb{Z}G$-module.*

*Proof.* It is clear that $\Omega$ is an abelian group with respect to multiplication. Indeed, $1 \in \Omega$ since $K/k$ is abelian by assumption, and if $\alpha \in \Omega$ then $\alpha^{-1} \in \Omega$ since $K(\alpha^{1/w_K}) = K(\alpha^{-1/w_K})$ so $K(\alpha^{-1/w_K})/k$ is abelian. Moreover, if $\beta \in \Omega$ then the extension $K((\alpha\beta)^{1/w_K})/k$ is abelian since it is contained in the compositum of two abelian extensions, so $\alpha\beta \in \Omega$.

It remains to verify that $\Omega$ is $G$-equivariant. That is, we need to show that if $\sigma \in G$ and $\alpha \in K$ is $w_K$-abelian, then $\sigma(\alpha)$ is $w_K$-abelian. To do this, first choose $a \in K$ such that

$$a^{w_K} = \alpha.$$

Thus, $K(\alpha^{1/w_K}) = K(a)$. Then, choose a lift of $\sigma$ to $K(a)$ and denote this lift by $\widetilde{\sigma}$. Then we have

$$\sigma(\alpha) = \widetilde{\sigma}(a^{w_K}) = \widetilde{\sigma}(a)^{w_K}.$$

Thus,

$$K(\sigma(a)^{1/w_K}) = K(\widetilde{\sigma}(a)) \subseteq K(a),$$

where the last step follows since $K(a)/k$ is Galois. This shows that $\sigma(a)$ is $w_K$-abelian, completing the proof. □

It was remarked previously that Conjecture 2.1 can be referred to by $St(K/k, S, v_0)$, with no mention of the place $w_0$ lying above $v_0$. This is justified with the following lemma.

**Lemma 2.4.** *The truth of $St(K/k, S, v_0, w_0)$ is independent of the choice of $w_0$ above $v_0$.*

*Proof.* Assume that $St(K/k, S, v_0, w_0)$ holds and $\varepsilon$ is a Stark unit. Assume that $w_0'$ is another place of $K$ that lies above $v_0$. Then there exists a $\tau \in G$ such that $w_0' = w_0^\tau$. For all $\chi \in \widehat{G}$ we have

$$\begin{aligned} L'_{K/k,S}(0,\chi) &= -\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^\sigma|_{w_0} \\ &= -\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^\sigma|_{{w_0'}^{\tau^{-1}}} \\ &= -\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^{\tau\sigma}|_{w_0'} \\ &= -\frac{1}{w_K}\sum_{\sigma\in G}\chi(\sigma)\log|(\varepsilon^{\tau})^\sigma|_{w_0'}, \end{aligned}$$

where the last equality holds because $G$ is abelian.

To verify the absolute value conditions, first assume $|S| \geq 3$ and $w \nmid v_0$. Then $|\varepsilon|_w = 1$ so

$$|\varepsilon^\tau|_w = |\varepsilon|_{w^{\tau^{-1}}} = 1,$$

since $w^{\tau^{-1}} \nmid v_0$.

Next assume $|S| = 2$ and let $\sigma \in G$. Then $|\varepsilon|_w = |\varepsilon|_{w^\sigma}$ so

$$|\varepsilon^\tau|_{w^\sigma} = |\varepsilon|_{(w^\sigma)^{\tau^{-1}}} = |\varepsilon|_{(w^{\tau^{-1}})^\sigma} = |\varepsilon|_{w^{\tau^{-1}}} = |\varepsilon^\tau|_w,$$

since $w^{\tau^{-1}} \nmid v_0$. This verifies the absolute value conditions. Moreover, $\varepsilon^\tau$ is $w_K$-abelian by Lemma 2.3. Thus, $\varepsilon^\tau$ is a Stark unit for $St(K/k, S, v_0, w_0^\tau)$. □

Under the assumptions of Conjecture 2.1 it is guaranteed that $s = 0$ is a zero of at least order 1 for the $L$-function in (2.1), as the following lemma shows.

**Lemma 2.5.** *If the extension $K/k$ and the set of primes $S$ satisfy the hypotheses in Conjecture 2.1 then* $\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi) \geq 1$ *for all* $\chi \in \widehat{G}$.

*Proof.* Because $K/k$ is abelian we may use Theorem 1.19 to compute the order of vanishing. Hypothesis (2) of Conjecture 2.1 ensures that $|S| - 1 \geq 1$ so the order of vanishing is at least 1 when $\chi = \chi_1$. In addition, hypothesis (3) of Conjecture 2.1 ensures that $S$ contains a prime $v$ which splits completely in $K$. In this case $G_v = 1$ so it is clear that $G_v \subseteq \ker(\chi)$. Thus,

$$|\{v \in S \mid G_v \subseteq \ker(\chi)\}| \geq 1,$$

meaning that the order of vanishing is at least 1 when $\chi \neq \chi_1$. □

Though the set $S$ can be any finite set of places containing the infinite places and the ramified places, it suffices to prove $St(K/k, S, v_0)$ for the minimal set $S$, as shown in the following lemma.

**Lemma 2.6.** *Let $S' \supset S$. If $St(K/k, S, v_0)$ is true then $St(K/k, S', v_0)$ is true.*

*Proof.* Assume that $St(K/k, S, v_0)$ holds and $\varepsilon$ is a Stark unit. By assumption $S$ contains the infinite places and the ramified places, so $S'$ does also. It suffices, then, to prove the lemma for $S' = S \cup \{\mathfrak{p}\}$ for some unramified prime $\mathfrak{p}$ of $k$. Then, using the definition of the $L$-function as a product over prime ideals we have

$$L_{K/k,S'}(s, \chi) = L_{K/k,S}(s, \chi) \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right).$$

Taking the derivative, evaluating at $s = 0$, and using Lemma 2.5 to note that $L_{K/k,S'}(0, \chi) = 0$ we have

$$\begin{aligned} L'_{K/k,S'}(0, \chi) &= L'_{K/k,S}(0, \chi)(1 - \chi(\sigma_{\mathfrak{p}})) \\ &= -\frac{1}{w_K} \left( \sum_{\sigma} \chi(\sigma) \log |\varepsilon^{\sigma}|_{w_0} - \sum_{\sigma} \chi(\sigma_{\mathfrak{p}}\sigma) \log |\varepsilon^{\sigma}|_{w_0} \right). \end{aligned}$$

We now make a change of variables in the second summation, letting $\tau = \sigma_{\mathfrak{p}}\sigma$. As $\sigma$ runs over all elements of $G$ so does $\tau$. Thus, we have

$$\sum_{\sigma \in G} \chi(\sigma_{\mathfrak{p}}\sigma) \log |\varepsilon^{\sigma}|_{w_0} = \sum_{\tau \in G} \chi(\tau) \log |\varepsilon^{\sigma_{\mathfrak{p}}^{-1}\tau}|_{w_0}.$$

This implies that

$$\begin{aligned} L'_{K/k,S'}(0,\chi) &= -\frac{1}{w_K}\left(\sum_\sigma \chi(\sigma)\log|\varepsilon^\sigma|_{w_0} - \sum_\sigma \chi(\sigma)\log|\varepsilon^{\sigma_\mathfrak{p}^{-1}\sigma}|_{w_0}\right) \\ &= -\frac{1}{w_K}\sum_\sigma \chi(\sigma)\log\left|\left(\frac{\varepsilon}{\varepsilon^{\sigma_\mathfrak{p}^{-1}}}\right)^\sigma\right|_{w_0} \\ &= -\frac{1}{w_K}\sum_\sigma \chi(\sigma)\log|(\varepsilon^{1-\sigma_\mathfrak{p}^{-1}})^\sigma|_{w_0}. \end{aligned}$$

Thus, to prove the lemma it suffices to show that $\varepsilon' := \varepsilon^{1-\sigma_\mathfrak{p}^{-1}}$ is a Stark unit for $St(K/k, S', v_0)$. We note first that $\varepsilon'$ is $w_K$-abelian over $k$ by Lemma 2.3. It remains to show that $\varepsilon'$ is an $S'_K$-unit and that it has the proper absolute values for places not lying above $v_0$.

To verify that $\varepsilon'$ is an $S'_K$-unit it suffices to show that it is an $S_K$-unit since $S_K \subseteq S'_K$. Let $w$ be a place of $K$ that does not lie above any of the places in $S$. We need to show that $|\varepsilon'|_w = 1$. Because $\varepsilon$ is a Stark unit for $St(K/k, S, v_0)$ this means $|\varepsilon|_w = 1$. Moreover, for any place $v$ of $k$ the Galois group $G$ acts on the places lying above $v$. Given $\sigma \in G$ it follows that $w^\sigma$ does not lie above any of the places in $S$ so $|\varepsilon|_{w^{\sigma_\mathfrak{p}}} = 1$. Thus, we have

$$|\varepsilon'|_w = \frac{|\varepsilon|_w}{|\varepsilon^{\sigma_\mathfrak{p}^{-1}}|_w} = \frac{|\varepsilon|_w}{|\varepsilon|_{w^{\sigma_\mathfrak{p}}}} = 1, \tag{2.3}$$

meaning that $\varepsilon'$ is an $S_K$-unit. All that remains is to verify the absolute value conditions. Since $S' \supset S$ this means $|S'| \geq 3$, so we only need to consider this case. Given that $\varepsilon'$ is an $S_K$-unit, $|\varepsilon'|_w = 1$ for all places not in $S_K$. Thus, all that must be shown is that $|\varepsilon'|_w = 1$ for the places in $S_K$ which do not lie above $v_0$. However, this is immediate because $\varepsilon$ is a Stark unit, so (2.3) still holds when $w$ is any place of $S_K$ not lying above $v_0$. □

In Lemma 2.8 it will be shown that if $St(K/k, S, v_0)$ is true, then $St(F/k, S, v_0)$ is true for all intermediate extensions $F/k$. First, we need a result from Tate.

**Lemma 2.7.** *Let $K/k$ be a finite abelian extension of number fields, and let $\alpha \in K$.*

*Furthermore, let $m \in \mathbb{N}$ and $a \in \overline{K}$ such that $K(a) \subseteq K^{ab}$ and such that*

$$a^m = \alpha.$$

*Then, there exists $\beta \in K^\times$ which is $w_K$-abelian over $k$ and*

$$\alpha^{w_K} = \zeta\beta^m$$

*for some $\zeta \in \mu(K)$.*

*Proof.* This is a particular case of the implication (b) $\implies$ (a) of Proposition 1.2 in [22, Ch 6]. □

**Lemma 2.8.** *Let $k \subset F \subset K$ be a tower of number fields with $K/k$ abelian. If $St(K/k, S, v_0)$ is true then $St(F/k, S, v_0)$ is true.*

*Proof.* Suppose $St(K/k, S, v_0)$ is true and that $\varepsilon$ is a Stark unit. Given this, one easily verifies that the hypotheses of Conjecture 2.1 are satisfied by $S$ and $v_0$ for the extension $F/k$. Indeed, since $v_0$ splits completely in $K$ it must also split completely in $F$. Moreover, any prime which ramifies in $F$ must also ramify in $K$, so $S$ contains all the places which ramify in $F$ (and also potentially some which do not ramify).

Now, Lemma 2.7 predicts the existence of a particular $S_F$-unit $\varepsilon_F$ as follows. Let $\alpha = N_{K/F}(\varepsilon) \in F^\times$. Since $\varepsilon$ is $w_K$-abelian over $k$, this means $\alpha$ is as well by Lemma 2.3. Thus, Lemma 2.7 with $m = w_K$ implies that there exists $\varepsilon_F \in F^\times$ which is $w_F$-abelian over $k$ and such that

$$N_{K/F}(\varepsilon)^{w_F} = \zeta \cdot \varepsilon_F^{w_K} \tag{2.4}$$

for some $\zeta \in \mu(K)$. Moreover, since $\varepsilon$ is an $S_K$-unit, (2.4) shows immediately that $\varepsilon_F$ is an $S_F$-unit.

From this condition, one can verify that $\varepsilon_F$ is a Stark unit for $St(F/k, S, v_0)$. We begin by verifying (2.1). To do this, fix a place $w_0$ of $K$ lying over $v_0$, let $G = \mathrm{Gal}(K/k)$, let $H = \mathrm{Gal}(K/F)$, and let $\Gamma = \mathrm{Gal}(F/k)$. If $\pi : G \to \Gamma$ is the the map defined by $\sigma \mapsto \sigma|_F$, then given a character $\chi$ on $\Gamma$ we get a character $\widetilde{\chi}$ on $G$ by composing with $\pi$. From the inflation property of L-functions (see Proposition 10.4

in [14, Ch 7 §10]) it follows that $L_{F/k,S}(s,\chi) = L_{K/k,S}(s,\widetilde{\chi})$. Thus,

$$\begin{aligned} L'_{F/k,S}(0,\chi) &= L'_{K/k,S}(0,\widetilde{\chi}) \\ &= -\frac{1}{w_K}\sum_{\sigma\in G}\widetilde{\chi}(\sigma)\log|\varepsilon^{\sigma}|_{w_0}. \end{aligned}$$

For each $\gamma \in \Gamma$ choose a lift $\sigma_\gamma \in G$. Galois theory tells us $\Gamma \cong G/H$ via the map $\gamma \mapsto \sigma_\gamma$. Thus, $\sigma_\gamma\tau$ runs over $G$ as $\gamma$ runs over $\Gamma$ and $\tau$ runs over $H$. Using this produces

$$\begin{aligned} L'_{F/k,S}(0,\chi) &= -\frac{1}{w_K}\sum_{\gamma\in\Gamma}\sum_{\tau\in H}\widetilde{\chi}(\sigma_\gamma\tau)\log|\varepsilon^{\sigma_\gamma\tau}|_{w_0} \\ &= -\frac{1}{w_K}\sum_{\gamma\in\Gamma}\chi(\gamma)\sum_{\tau\in H}\log|\varepsilon^{\sigma_\gamma\tau}|_{w_0} \\ &= -\frac{1}{w_K}\sum_{\gamma\in\Gamma}\chi(\gamma)\log|N_{K/F}(\varepsilon)^{\sigma_\gamma}|_{w_0}. \end{aligned}$$

Because $N_{K/F}(\varepsilon) \in F$ and $\sigma_\gamma|_F = \gamma$, we can use (2.4) to conclude

$$\begin{aligned} L'_{F/k,S}(0,\chi) &= -\frac{1}{w_K}\sum_{\gamma\in\Gamma}\chi(\gamma)\log|N_{K/F}(\varepsilon)^{\gamma}|_{w_0} \\ &= -\frac{1}{w_F w_K}\sum_{\gamma\in\Gamma}\chi(\gamma)\log|(\varepsilon_F^{w_K})^{\gamma}|_{w_0} \qquad (2.5) \\ &= -\frac{1}{w_F}\sum_{\gamma\in\Gamma}\chi(\gamma)\log|(\varepsilon_F)^{\gamma}|_{w_0}, \end{aligned}$$

which verifies (2.1). To prove that the unit $\varepsilon_F$ is a Stark unit for $St(F/k,S,v_0)$, it remains to verify the absolute value conditions. First, assume that $|S| \geq 3$ and let $w$

be a place of $K$ not lying above $v_0$. We have

$$\begin{aligned} |\varepsilon_F|_w^{w_K} &= |(\varepsilon_F)^{w_K}|_w \\ &= |\zeta \cdot N_{K/F}(\varepsilon)|_w^{w_F} \\ &= |N_{K/F}(\varepsilon)|_w^{w_F} \\ &= \prod_{\tau \in H} |\varepsilon^\tau|_w^{w_F} \\ &= \prod_{\tau \in H} |\varepsilon|_{w^{\tau^{-1}}}^{w_F} \\ &= 1. \end{aligned}$$

The last equality holds since for each $\tau \in H$ the place $w^{\tau^{-1}} \nmid v_0$. Thus, by assumption, $|\varepsilon|_{w^{\tau^{-1}}} = 1$ for all $\sigma \in H$. Moreover, each place $\widetilde{w} \nmid v_0$ of $F$ is equivalent to the restriction to $F$ of some place $w \nmid v_0$ of $K$. Thus, it follows that $|\varepsilon_F|_{\widetilde{w}} = 1$ for all places $\widetilde{w} \nmid v_0$ of $F$.

Now, assume that $|S| = 2$. Let $\sigma \in G$. Again, $H = \mathrm{Gal}(K/F)$ and $w$ denotes a place of $K$ not lying above $v_0$. Then we have

$$\begin{aligned} |\varepsilon_F|_{w^\sigma}^{w_K} &= |N_{K/F}(\varepsilon)|_{w^\sigma}^{w_F} \\ &= \prod_{\tau \in H} |\varepsilon|_{w^{\tau^{-1}\sigma}}^{w_F} \\ &= \prod_{\tau \in H} |\varepsilon|_{w^{\tau^{-1}}}^{w_F} \\ &= |\varepsilon_F|_w^{w_K}. \end{aligned} \tag{2.6}$$

The third line above follows from the fact that $G$ is abelian as well as the assumption that $|\varepsilon|_w = |\varepsilon|_{w^\sigma}$ for all $\sigma \in G$. Thus, it has been shown that

$$|\varepsilon_F|_{w^\sigma} = |\varepsilon_F|_w$$

for all $\sigma \in G$ and for all places $w$ of $K$ not lying above $v_0$. It must be shown that this expression still holds for the places of $F$ in $S_F$ not lying above $v_0$, and for elements of the Galois group $H$.

Let $\widetilde{w}$ be a place of $F$ such $\widetilde{w} \nmid v_0$, let $\tau \in H$, and let $\sigma \in G$ be an extension of $\tau$. Because $\widetilde{w}$ is equivalent to a place $w$ of $K$ not lying above $v$, there exists a $c \in \mathbb{R}$

such that $|\cdot|_{\widetilde{w}} = |\cdot|_w^c$. Thus,

$$|\varepsilon_F|_{\widetilde{w}^\tau} = |\varepsilon_F|_{\widetilde{w}^\sigma} = |\varepsilon_F|_{w^\sigma}^c = |\varepsilon_F|_w^c = |\varepsilon_F|_{\widetilde{w}},$$

which is the desired result. $\square$

# 3 Kronecker's Second Limit Formula

The goal of this chapter is to prove Kronecker's second limit formula and apply it to produce an expression for the derivative of the $S$-truncated L-function evaluated at $s = 0$. This expression (3.31) will ultimately be used to verify (2.1) of Stark's conjecture in the case when $k$ is quadratic imaginary.

We begin with a statement of Kronecker's second limit formula, followed by an explanation of its relationship with the $S$-truncated L-function of Stark's conjecture. The proof will make use of a few standard Fourier series, the gamma function, and some properties of the modified Bessel function of the second kind. An exposition of each of these topics are also included for clarity. For the proof itself, we follow Lang [12], expanding on the exposition throughout.

## 3.1 The Statement of the Theorem

Given $z \in \mathbb{C}$ define $q_z := e^{2\pi i z}$. Let $u, v \in \mathbb{R}$, and let $\tau = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. Define

$$E_{u,v}(\tau, s) = \sum_{(m,n)}{}' e^{2\pi i(mu+nv)} \frac{y^s}{|m\tau + n|^{2s}}, \tag{3.1}$$

where the sum is taken over all nonzero pairs $(m, n) \in \mathbb{Z}^2$. For fixed $u, v$, and $\tau$ we may consider $E_{u,v}(\tau, s)$ as a function of $s$. This function converges absolutely and uniformly on compact subsets of $\mathbb{C}$ when $\mathrm{Re}(s) > 1$.

It will be shown that $E_{u,v}(\tau, s)$ can be analytically extended to include the point $s = 1$. Kronecker's first and second limit formulas provide expressions for $E_{u,v}(\tau, 1)$ in different cases. If $u$ and $v$ are both integers then $e^{2\pi i(mu+nv)} = 1$ and this case is considered by Kronecker's first limit formula. The case in which $u$ and $v$ are not both integers is considered by Kronecker's second limit formula.

We now must also introduce the product expansion of the Siegel function. Another definition of this function as well as a discussion of its important properties are provided in Section 4.6. For the present task we need not concern ourselves with any properties that are not deducible from the product expansion. Given $(t_1, t_2) \in \mathbb{R}$ and $\tau \in \mathbb{H}$, let $z = t_1\tau + t_2$. Define

$$g_{t_1,t_2}(\tau) = -q_\tau^{\frac{1}{2}(t_1^2 - t_1 + \frac{1}{6})} e^{\pi i t_2 (t_1 - 1)} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}).$$

We are now able to state the theorem to be proven.

**Theorem 3.1** (Kronecker's Second Limit Formula)**.** *Let $u, v \in \mathbb{R}$ be such that $u$ and $v$ are not both integers. The function $E_{u,v}(\tau, s)$ can be analytically continued to include $s = 1$ and*

$$E_{u,v}(\tau, 1) = -2\pi \log |g_{-v,u}(\tau)| . \tag{3.2}$$

This theorem will be proven later in Section 3.6. First, we concern ourselves with discussing the relationship between $E_{u,v}(s, \tau)$ and the L-function associated to the ray class group. This will serve to motivate the importance of Theorem 3.1.

## 3.2 L-functions and the Series $E_{u,v}(\tau, s)$

In what follows, let $k$ be a quadratic imaginary number field, $\mathfrak{m} \neq (1)$ be an integral ideal of $k$, and $w_k$ be the number of roots of unity in $k$. Recall that since $k$ has no real embeddings in this setup, all elements of $k$ are totally positive. Thus, $Cl_{\mathfrak{m}}^{res}(k) = Cl_{\mathfrak{m}}(k)$. Throughout this section we assume familiarity with the notation regarding characters and Gauss sums introduced in Section 1.4.

**Lemma 3.2.** *Let $\psi$ be a proper character of the ray class group $Cl_{\mathfrak{m}}(k)$. Fix $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ such that $(\mathfrak{m}, y\mathfrak{m}\mathfrak{d}) = 1$. For each class $C$ in the ordinary class group $Cl(k)$ fix an integral ideal $\mathfrak{b}_C \in C^{-1}$ that is prime to $\mathfrak{m}$. Then*

$$L_{\mathfrak{m}}(s, \psi) = \frac{1}{w_k \tau(\overline{\psi}_f, y)} \sum_{C \in Cl(k)} \overline{\psi}(\mathfrak{b}_C) \mathbb{N}(\mathfrak{b}_C)^s \sum_{\xi \in \mathfrak{b}_C} \frac{\tau(\overline{\psi}_f, \xi y)}{N_{k/\mathbb{Q}}(\xi)^s},$$

*where $\psi_f$ was defined in Section 1.6.*

*Proof.* Recall that by definition

$$L_{\mathfrak{m}}(s,\psi) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \frac{\psi(\mathfrak{a})}{\mathbb{N}(\mathfrak{a})^s},$$

where the sum ranges over integral ideals $\mathfrak{a}$ prime to $\mathfrak{m}$. Splitting the sum according to classes in the class group $Cl(k)$ produces

$$L_{\mathfrak{m}}(s,\psi) = \sum_{C\in Cl(k)} \sum_{\substack{\mathfrak{a}\in C\\(\mathfrak{a},\mathfrak{m})=1}} \frac{\psi(\mathfrak{a})}{\mathbb{N}(\mathfrak{a})^s}. \tag{3.3}$$

Given an integral ideal $\mathfrak{a} \in C$ with $(\mathfrak{a},\mathfrak{m}) = 1$ the integral ideal $\mathfrak{a}\mathfrak{b}_C$ is principal and relatively prime to $\mathfrak{m}$. Thus, we may write

$$\mathfrak{a}\mathfrak{b}_C = (\xi_{\mathfrak{a}})$$

for some $\xi_{\mathfrak{a}} \in \mathcal{O}_k$. Conversely, given an arbitrary integral principal ideal $(\xi) \subseteq \mathfrak{b}_C$ with $(\xi,\mathfrak{m}) = 1$ the ideal $\mathfrak{a}_\xi \in C$ defined by

$$\mathfrak{a}_\xi = (\xi)\mathfrak{b}_C^{-1}$$

is integral and relatively prime to $\mathfrak{m}$. The maps $\mathfrak{a} \mapsto (\xi_{\mathfrak{a}})$ and $(\xi) \mapsto \mathfrak{a}_\xi$ are inverses of each other, and thus we have established a bijection between the set of integral ideals $\mathfrak{a} \in C$ with $(\mathfrak{a},\mathfrak{m}) = 1$ and the set of principal integral ideals $(\xi) \subseteq \mathfrak{b}_C$ with $(\xi,\mathfrak{m}) = 1$.

Using this correspondence, the inner sum in (3.3) may be rewritten as a sum over the ideals $(\xi_{\mathfrak{a}})$. For each $\mathfrak{a} \in C$ we have

$$\psi(\mathfrak{a}) = \psi(\mathfrak{b}_C^{-1}(\xi_{\mathfrak{a}})) = \overline{\psi}(\mathfrak{b}_C)\psi_f(\xi_{\mathfrak{a}}).$$

Furthermore, a standard fact from algebraic number theory says that $\mathbb{N}((\xi)) = |N_{k/\mathbb{Q}}(\xi)|$. However, since $k$ is quadratic imaginary $N_{k/\mathbb{Q}}(\xi) \geq 0$ and we may write

$\mathbb{N}((\xi)) = N_{k/\mathbb{Q}}(\xi)$, which implies

$$\mathbb{N}(\mathfrak{a}) = \mathbb{N}(\mathfrak{b}_C^{-1}(\xi_\mathfrak{a})) = \frac{N_{k/\mathbb{Q}}(\xi_\mathfrak{a})}{\mathbb{N}(\mathfrak{b}_C)}.$$

Together, these results produce the expression

$$L_\mathfrak{m}(s,\psi) = \sum_{C\in Cl(k)} \overline{\psi}(\mathfrak{b}_C)\mathbb{N}(\mathfrak{b}_C)^s \sum_{\substack{(\xi_\mathfrak{a})\subseteq \mathfrak{b}_C \\ (\xi_\mathfrak{a},\mathfrak{m})=1}} \frac{\psi_f(\xi_\mathfrak{a})}{N_{k/\mathbb{Q}}(\xi_\mathfrak{a})^s}.$$

Recall that two integral ideals $(\xi_1)$ and $(\xi_2)$ are equal if and only if there exists a unit $\varepsilon \in \mathcal{O}_k$ with $\xi_2 = \varepsilon\xi_1$. Since $k$ is quadratic imaginary, the only units are the roots of unity. Thus, letting $w_k$ be the number of roots of unity in $k$, it follows that

$$L_\mathfrak{m}(s,\psi) = \frac{1}{w_k} \sum_{C\in Cl(k)} \overline{\psi}(\mathfrak{b}_C)\mathbb{N}(\mathfrak{b}_C)^s \sum_{\substack{\xi\in \mathfrak{b}_C \\ (\xi,\mathfrak{m})=1}} \frac{\psi_f(\xi)}{N_{k/\mathbb{Q}}(\xi)^s}.$$

We will now write $\psi_f(\xi)$ in terms of Gauss sums to achieve the desired result. From Lemmas 1.9 and 1.10 it follows immediately that

$$\frac{\tau(\overline{\psi}_f, \xi y)}{\tau(\overline{\psi}_f, y)} = \begin{cases} \psi_f(\xi), & \text{if } (\xi,\mathfrak{m}) = 1; \\ 0, & \text{if } (\xi,\mathfrak{m}) \neq 1. \end{cases}$$

Thus, we have

$$L_\mathfrak{m}(s,\psi) = \frac{1}{w_k \tau(\overline{\psi}_f, y)} \sum_{C\in Cl(k)} \overline{\psi}(\mathfrak{b}_C)\mathbb{N}(\mathfrak{b}_C)^s \sum_{\xi\in \mathfrak{b}_C} \frac{\tau(\overline{\psi}_f, \xi y)}{N_{k/\mathbb{Q}}(\xi)^s},$$

as desired. □

**Lemma 3.3.** *Let $\mathfrak{a} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ be a fractional ideal of the quadratic imaginary number field $k$, with $\tau = \omega_1/\omega_2 \in \mathbb{H}$. If $\lambda \in \mathfrak{a}$ then $\lambda = m\omega_1 + n\omega_2$ for unique $m, n \in \mathbb{Z}$, and the following expression holds:*

$$N_{k/\mathbb{Q}}(\lambda) = \frac{\mathbb{N}(\mathfrak{a})\sqrt{|d_k|}}{2\mathrm{Im}(\tau)} \cdot |m\tau + n|^2. \tag{3.4}$$

*Proof.* In what follows, for $x \in k$ write $N(x) := N_{k/\mathbb{Q}}(x)$. Each $\lambda \in \mathfrak{a}$ can be written uniquely as

$$\lambda = m\omega_1 + n\omega_2 = \omega_2(m\tau + n).$$

Then, taking the norm of both sides produces

$$N(\lambda) = N(\omega_2)N(m\tau + n) = N(\omega_2)|m\tau + n|^2. \tag{3.5}$$

Moreover, a standard fact from algebraic number theory gives

$$d(\mathfrak{a}) = \mathbb{N}(\mathfrak{a})^2 d_k. \tag{3.6}$$

The discriminant on the left can be calculated in terms of $\omega_2$ and $\tau$ as

$$\begin{aligned} d(\mathfrak{a}) &= \det \begin{pmatrix} \omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2 \end{pmatrix}^2 \\ &= (\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1)^2 \\ &= (\omega_2\overline{\omega}_2(\tau - \overline{\tau}))^2 \\ &= \mathbb{N}(\omega_2)^2 \cdot (2i\mathrm{Im}(\tau))^2. \end{aligned}$$

Using this along with (3.5) and (3.6) gives the desired result. □

**Theorem 3.4.** *Let $\psi$ be a proper character of the ray class group $Cl_{\mathfrak{m}}(k)$. Fix $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ such that $(\mathfrak{m}, y\mathfrak{m}\mathfrak{d}) = 1$. For each class $R \in Cl_{\mathfrak{m}}(k)$ fix an integral ideal $\mathfrak{b}_R \in R$ and choose $\omega_1, \omega_2$ such that $\tau_R := \omega_1/\omega_2 \in \mathbb{H}$ and $y\mathfrak{b}_R = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Then, letting $\mathfrak{a} = y\mathfrak{m}\mathfrak{d}$*

$$L_{\mathfrak{m}}(s, \psi) = \frac{2^s\psi(\mathfrak{a})}{w_{\mathfrak{m}}\tau(\overline{\psi}_f, y)\sqrt{|d_k|}^s} \sum_{R \in Cl_{\mathfrak{m}}(k)} \overline{\psi}(\mathfrak{a}\mathfrak{b}_R)E_{u(\mathfrak{b}_R),v(\mathfrak{b}_R)}(\tau_R, s),$$

*where $u(\mathfrak{b}_R) = \mathrm{Tr}(\omega_1)$, $v(\mathfrak{b}_R) = \mathrm{Tr}(\omega_2)$, and $w_{\mathfrak{m}}$ denotes the number of roots of unity in $k$ which are congruent to 1 modulo $\mathfrak{m}$.*

*Proof.* Begin by using the Definition 1.7 of the Gauss sum to expand $\tau(\overline{\psi}_f, \xi y)$ in the result from Lemma 3.2. This gives

$$\begin{aligned}
L_{\mathfrak{m}}(s,\psi) &= \frac{1}{w_k \tau(\overline{\psi}_f, y)} \sum_{C \in Cl(k)} \overline{\psi}(\mathfrak{b}_C)\mathbb{N}(\mathfrak{b}_C)^s \sum_{\xi \in \mathfrak{b}_C} \frac{\tau(\overline{\psi}_f, \xi y)}{N_{k/\mathbb{Q}}(\xi)^s} \\
&= \frac{1}{w_k \tau(\overline{\psi}_f, y)} \sum_{C \in Cl(k)} \overline{\psi}(\mathfrak{b}_C)\mathbb{N}(\mathfrak{b}_C)^s \sum_{\xi \in \mathfrak{b}_C} \sum_{z \bmod \mathfrak{m}} \frac{\overline{\psi}_f(z) e^{2\pi i \mathrm{Tr}(z\xi y)}}{N_{k/\mathbb{Q}}(\xi)^s} \\
&= \frac{1}{w_k \tau(\overline{\psi}_f, y)} \sum_{C \in Cl(k)} \sum_{z \bmod \mathfrak{m}} \overline{\psi}(z\mathfrak{b}_C)\mathbb{N}(z\mathfrak{b}_C)^s \sum_{\xi \in \mathfrak{b}_C} \frac{e^{2\pi i \mathrm{Tr}(z\xi y)}}{N_{k/\mathbb{Q}}(z\xi)^s}.
\end{aligned}$$

We will now examine the relationship between the elements of $(\mathcal{O}_k/\mathfrak{m})^\times, Cl(k)$, and $Cl_{\mathfrak{m}}(k)$. Familiarity with the notation introduced in Section 1.5 is assumed. Since $k$ is quadratic imaginary, its only units are the roots of unity. Thus, letting $\mu(k)$ be the set of roots of unity of $k$ and letting $\mu_{\mathfrak{m}}(k)$ be the set of roots of unity of $k$ congruent to 1 modulo $\mathfrak{m}$, the exact sequence from Theorem 1.11 can be written as

$$1 \longrightarrow \mu_{\mathfrak{m}}(k) \longrightarrow \mu(k) \xrightarrow{\rho} (\mathcal{O}_k/\mathfrak{m})^\times \xrightarrow{\phi} Cl_{\mathfrak{m}}(k) \longrightarrow Cl(k) \longrightarrow 1.$$

Letting $M := \mu(k)/\mu_{\mathfrak{m}}(k)$ we may convert this sequence to the following short exact sequence

$$1 \longrightarrow (\mathcal{O}_k/\mathfrak{m})^\times/M \xrightarrow{\phi} Cl_{\mathfrak{m}}(k) \xrightarrow{\pi} Cl(k) \longrightarrow 1. \tag{3.7}$$

This sequence can be used to show that the elements $z\mathfrak{b}_C$ in the summation above run over the elements of $Cl_{\mathfrak{m}}(k)$ exactly $w_k/w_{\mathfrak{m}}$ times. To see why this is, consider first an arbitrary short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Furthermore, let $\iota : C \to B$ be a function such that $\beta \circ \iota = \mathrm{id}_C$. Since $C \cong B/A$ this means $\{\iota(c) \mid c \in C\}$ is a complete set of representatives of $B/A$ and

$$B = \{\alpha(a) \cdot \iota(c) \mid a \in A, c \in C\}.$$

Now, apply this result to the sequence in (3.7), letting $\iota$ be the function sending $\mathfrak{b}_C$ to its ideal class in $Cl_{\mathfrak{m}}(k)$. This implies that $z\mathfrak{b}_C$ runs exactly once over $Cl_{\mathfrak{m}}(k)$ as $z$ runs over a set of representatives of $(\mathcal{O}_k/\mathfrak{m})^\times/M$ and $C$ runs over $Cl(k)$. So, if instead

$z$ runs over all of $(\mathcal{O}_k/\mathfrak{m})^\times$ this means $z\mathfrak{b}_C$ runs over $Cl_\mathfrak{m}(k)$ exactly $|M| = w_k/w_\mathfrak{m}$ times. Applying this to the sum above we have

$$L_\mathfrak{m}(s,\psi) = \frac{1}{w_\mathfrak{m}\tau(\overline{\psi}_f, y)} \sum_{R\in Cl_\mathfrak{m}(k)} \overline{\psi}(\mathfrak{b}_R)\mathbb{N}(\mathfrak{b}_R)^s \sum_{\xi\in\mathfrak{b}_R} \frac{e^{2\pi i \mathrm{Tr}(\xi y)}}{N_{k/\mathbb{Q}}(\xi)^s}.$$

Now make a change of variables, letting $\lambda = \xi y$. As $\xi$ runs over $\mathfrak{b}_R$, $\lambda$ runs over $y\mathfrak{b}_R$. Thus,

$$L_\mathfrak{m}(s,\psi) = \frac{1}{w_\mathfrak{m}\tau(\overline{\psi}_f, y)} \sum_{R\in Cl_\mathfrak{m}(k)} \overline{\psi}(\mathfrak{b}_R)\mathbb{N}(y\mathfrak{b}_R)^s \sum_{\lambda\in y\mathfrak{b}_R} \frac{e^{2\pi i \mathrm{Tr}(\lambda)}}{N_{k/\mathbb{Q}}(\lambda)^s}.$$

Now, choose $\omega_1, \omega_2$ such that $\tau_R := \omega_1/\omega_2 \in \mathbb{H}$ and $y\mathfrak{b}_R = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Noting that each $\lambda \in y\mathfrak{b}_R$ can be represented uniquely as $\lambda = m\omega_1 + n\omega_2$ for some $m, n \in \mathbb{Z}$, and applying (3.4) with $\mathfrak{a} = y\mathfrak{b}_R$ produces

$$\begin{aligned} L_\mathfrak{m}(s,\psi) &= \frac{2^s}{w_\mathfrak{m}\tau(\overline{\psi}_f, y)\sqrt{|d_k|}^s} \sum_{R\in Cl_\mathfrak{m}(k)} \overline{\psi}(\mathfrak{b}_R) \sum_{(m,n)}{}' e^{2\pi i \mathrm{Tr}(m\omega_1+n\omega_2)} \frac{\mathrm{Im}(\tau_R)^s}{|m\tau+n|^{2s}} \\ &= \frac{2^s}{w_\mathfrak{m}\tau(\overline{\psi}_f, y)\sqrt{|d_k|}^s} \sum_{R\in Cl_\mathfrak{m}(k)} \overline{\psi}(\mathfrak{b}_R) E_{\mathrm{Tr}(\omega_1),\mathrm{Tr}(\omega_2)}(\tau_R, s). \end{aligned}$$

Noting that

$$\overline{\psi}(\mathfrak{b}_R) = \overline{\psi}(\mathfrak{a}^{-1}\mathfrak{a}\mathfrak{b}_R) = \psi(\mathfrak{a})\overline{\psi}(\mathfrak{a}\mathfrak{b}_R)$$

produces the desired result. $\square$

With this motivation completed, we now turn toward an exposition of some background necessary for the proof of Theorem 3.1.

## 3.3 Fourier Series

This section briefly provides basic definitions and statements regarding Fourier series and their convergence. For proofs see [21, Ch. 5.4].

Consider a function $f : (a, b) \to \mathbb{R}$ defined on an open interval $(a, b)$ of length $L$. We define the periodic extension of $f$ by

$$f_{\mathrm{per}}(x) = f(x - Ln) \quad \text{for } x \in (a + Ln, b + Ln) \tag{3.8}$$

for all $n \in \mathbb{Z}$. Note that $f_{\text{per}}(x)$ is not defined when $x = a + Ln$ or when $x = b + Ln$. Given a function $g : [a, b] \to \mathbb{R}$ defined on a closed interval we will slightly abuse notation and refer to the periodic extension of the restriction $g|_{(a,b)}$ as $g_{\text{per}}(x)$.

If $f$ is a real valued function defined on a closed interval $[a, b]$ of length $L$, then the complex Fourier series of $f$ is given by

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$$

where

$$c_n = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}dx.$$

There are various criteria for the convergence of Fourier series. The one outlined in the following theorem will be sufficient for all that follows. We use the notation $f(c+) := \lim_{x\to c^+} f(x)$ and $f(c-) := \lim_{x\to c^-} f(x)$.

**Theorem 3.5.** *Let $f$ be a real valued function such that $f$ and $f'$ are piecewise continuous on an interval $[a, b]$ of length $L$. Then $S_f(x)$ converges pointwise to $\frac{1}{2}[f_{\text{per}}(x+) + f_{\text{per}}(x-)]$ for all $x \in \mathbb{R}$.*

*In particular, $S_f(x)$ converges pointwise to $\frac{1}{2}[f(x+) + f(x-)]$ for all $x \in (a, b)$.*

Below, a few Fourier series on the interval $[0, 1]$ are explicitly calculated. These will be needed later.

*Example 1:* Fix $c \in \mathbb{R}$ and consider the constant function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = c$. Then

$$c_n = \int_0^1 ce^{-2\pi inx}dx = \begin{cases} 0, & n \neq 0; \\ c, & n = 0, \end{cases}$$

so the Fourier series of $f(x)$ is simply given by $S_f(x) = c$.

*Example 2:* Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x$. A straightforward integration by parts gives

$$c_n = \int_0^1 xe^{-2\pi inx}dx = \begin{cases} \frac{1}{2}, & n = 0; \\ \frac{i}{2\pi n}, & n \neq 0. \end{cases}$$

Thus, the Fourier series of $f(x) = x$ is given by

$$x = \frac{1}{2} + \sum_{n\in\mathbb{Z}}{}' \frac{i}{2\pi n} e^{2\pi i n x}.$$

By Theorem 3.5 this series converges to $f(x) = x$ for all $x \in (0,1)$ and converges to $\frac{1}{2}$ for $x = 0$ and $x = 1$.

*Example 3:* Finally, let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$. This time, using integration by parts twice gives

$$c_n = \int_0^1 x^2 e^{-2\pi i n x} dx = \begin{cases} \frac{1}{3}, & n = 0; \\ \frac{i}{2\pi n} + \frac{1}{2\pi^2 n^2}, & n \neq 0. \end{cases}$$

This means the Fourier series of $f(x) = x^2$ is given by

$$x^2 = \frac{1}{3} + \sum_{n\in\mathbb{Z}}{}' \left( \frac{i}{2\pi n} + \frac{1}{2\pi^2 n^2} \right) e^{2\pi i n x}.$$

By Theorem 3.5 this series converges to $f(x) = x^2$ for all $x \in (0,1)$ and converges to $\frac{1}{2}$ for $x = 0$ and $x = 1$.

## 3.4 The Gamma Function and the Function $K_s(x)$

Recall from (1.15) that the gamma function is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \mathrm{Re}(s) > 0.$$

A useful relation which will be used later comes from performing a change of variables in this integral. Given $a \in \mathbb{R}_{>0}$, let $u = t/a$ in the definition of $\Gamma(s)$. This produces another expression for the gamma function

$$\Gamma(s) = a^s \int_0^\infty e^{-au} u^{s-1} du. \tag{3.9}$$

We will also make use of another function, which is similar to $\Gamma(s)$ but possesses a certain symmetry in $s$, which will be explained below. Let $a$ and $b$ be nonzero real numbers. Define the function

$$K_s(a,b) = \int_0^\infty e^{-(a^2t+b^2/t)} t^{s-1} dt. \tag{3.10}$$

For fixed pairs $a, b \in \mathbb{R} \setminus \{0\}$ the integral defining $K_s(a,b)$ converges absolutely for all $s \in \mathbb{C}$. Finally, to ease notation define for $x > 0$ the following function

$$K_s(x) = \int_0^\infty e^{-x(t+1/t)} t^{s-1} dt. \tag{3.11}$$

For fixed $x \in \mathbb{R}_{>0}$ the integral defining $K_s(x)$ converges absolutely for all $s \in \mathbb{C}$. By performing the change of variables $u = \left|\frac{a}{b}\right| t$ the function $K_s(a,b)$ can be written in terms of $K_s(x)$ as follows

$$\begin{aligned} K_s(a,b) &= \int_0^\infty e^{-(|ab|u+|ab|/u)} \left|\frac{b}{a}\right|^{s-1} u^{s-1} \left|\frac{b}{a}\right| du \\ &= \left|\frac{b}{a}\right|^s \int_0^\infty e^{-|ab|(u+1/u)} u^{s-1} du \\ &= \left|\frac{b}{a}\right|^s K_s(|ab|). \end{aligned} \tag{3.12}$$

As will be shown later in (3.26), the function $E_{u,v}(\tau)$ on the left side of Kronecker's limit formula can be rewritten in terms of an integral of the form $K_{\frac{1}{2}}(a,b)$. Thus, we will now evaluate this integral, first by finding an expression for $K_{\frac{1}{2}}(x)$ for $x > 0$ and then making use of the relation in (3.12).

By using the integral expression of $K_{-s}(x)$ and then performing a change of variables $u = t^{-1}$ it immediately follows that $K_s(x) = K_{-s}(x)$ as shown below.

$$\begin{aligned} K_{-s}(x) &= \int_0^\infty e^{-x(t+1/t)} t^{-s-1} dt \\ &= \int_0^\infty e^{-x(t+1/t)} \frac{t^{-s+1}}{t^2} dt \\ &= -\int_\infty^0 e^{-x(u+1/u)} u^{s-1} du \\ &= \int_0^\infty e^{-x(u+1/u)} u^{s-1} du = K_s(x). \end{aligned}$$

This is the useful symmetry in $s$ which was alluded to earlier. We will evaluate $K_{1/2}(x)$ by making use of this symmetry along with differentiation under the integral. First, in (3.11) make the change of variables $u = xt$. Since $x > 0$ this transforms the integral into

$$K_{1/2}(x) = \frac{1}{\sqrt{x}} \int_0^\infty e^{-(u+x^2/u)} u^{-1/2} du. \tag{3.13}$$

Now define the function $h : \mathbb{R}_{>0} \to \mathbb{R}$ by $h(x) = \sqrt{x} K_{1/2}(x)$. Differentiating under the integral sign produces

$$\frac{dh}{dx} = -2x \int_0^\infty e^{-(u+x^2/u)} u^{-3/2} du.$$

To transform this integral back into an integral of the form $K_s(x)$ perform the change of variables $t = \frac{u}{x}$ which is valid because $x > 0$. This produces

$$\begin{aligned} \frac{dh}{dx} &= -2x \int_0^\infty e^{-x(t+1/t)} (tx)^{-3/2} x dt \\ &= -2\sqrt{x} \int_0^\infty e^{-x(t+1/t)} t^{-3/2} dt \\ &= -2\sqrt{x} K_{-1/2}(x) = -2\sqrt{x} K_{1/2}(x) \\ &= -2h(x). \end{aligned}$$

This implies that $h(x) = Ae^{-2x}$ for some $A \in \mathbb{R}$. Recall that $h(x)$ was originally defined only for $x > 0$, so to calculate $A$ we use a limiting argument. Let $a, b, c \in \mathbb{R}_{>0}$ with $a \leq b \leq c$. Then from (3.13) we have

$$\begin{aligned} A &= \lim_{x\to 0} \int_0^\infty e^{-(u+x^2/u)} u^{-1/2} du \\ &= \lim_{x\to 0} \lim_{a\to 0} \int_a^b e^{-(u+x^2/u)} u^{-1/2} du + \lim_{x\to 0} \lim_{c\to\infty} \int_b^c e^{-(u+x^2/u)} u^{-1/2} du \\ &= \lim_{a\to 0} \int_a^b \lim_{x\to 0} e^{-(u+x^2/u)} u^{-1/2} du + \lim_{c\to\infty} \int_b^c \lim_{x\to 0} e^{-(u+x^2/u)} u^{-1/2} du \\ &= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{aligned}$$

In the process above we use the fact that the operator $\lim_{x\to 0}$ can commute with the other limits as well as the definite integrals. To justify this for the first term, let $F_a(x) = \int_a^b e^{-(u+x^2/u)} u^{-1/2} du$. Then it can be verified that $\sup_{x\in\mathbb{R}_{>0}} |F_a(x) - F_0(x)| \to 0$ as $a \to 0$ and that $\lim_{x\to 0} F_a(x)$ exists for all $a \in \mathbb{R}_{\geq 0}$. This allows the operators $\lim_{x\to 0}$ and $\lim_{a\to 0}$ to be swapped (see Theorem 1 in [24, Ch. 16.3.4]). In addition, defining $G_x(u) = e^{-(u+x^2/u)}u^{-1/2}$ it can be verified that $G_x(u)$ is integrable on $[a, b]$ for all $x \in \mathbb{R}_{>0}$ and that $\sup_{u\in[a,b]} |G_x(u) - G_0(u)| \to 0$ as $x \to 0$. This allows the operator $\lim_{x\to 0}$ to be swapped with the definite integral (see Theorem 3 in [24, Ch. 16.3.2]). Justification for the second term is very similar.

Using this along with (3.12) we find that for $a, b \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} K_{1/2}(a,b) &= \left|\frac{b}{a}\right|^{1/2} K_{1/2}(|ab|) \\ &= \left|\frac{b}{a}\right|^{1/2} \frac{h(|ab|)}{|ab|^{1/2}} \\ &= \frac{\sqrt{\pi}}{|a|}e^{-2|ab|}. \end{aligned} \tag{3.14}$$

In fact, this expression also holds when $b = 0$. This can be verified by using (3.9) and (3.10) to determine

$$\begin{aligned} K_{1/2}(a,0) &= \int_0^\infty e^{-a^2t}t^{-1/2}dt \\ &= \frac{(a^2)^{1/2}}{(a^2)^{1/2}} \int_0^\infty e^{-a^2t}t^{-1/2}dt \\ &= \frac{1}{|a|}\Gamma(1/2) \\ &= \frac{\sqrt{\pi}}{|a|}. \end{aligned}$$

This is the extent of what we will need to know about the formula $K_s(a, b)$.

## 3.5 The Fourier Transform and the Poisson Summation Formula

Given a function $f : U \to \mathbb{C}$ with $\mathbb{R} \subseteq U \subseteq \mathbb{C}$ define the Fourier transform of $f$ by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx. \tag{3.15}$$

Of course, we must discuss for which functions $f$ this integral converges. For this purpose we let $\mathfrak{F}$ denote the class of functions $f : U \to \mathbb{C}$ for which both of the following properties hold.

(a) There exists an $\alpha > 0$ for which $f(z)$ is holomorphic on the strip $\{z \in \mathbb{C} : |\text{Im}(z)| < \alpha\}$.

(b) There exists a $\beta > 0$ for which $|f(x+iy)| < \dfrac{\beta}{1+x^2}$ for all $x \in \mathbb{R}$ and $|y| < \alpha$.

If $f \in \mathfrak{F}$ then the integral in (3.15) converges and thus $\hat{f}(y)$ exists. Moreover, we have the following useful theorem.

**Theorem 3.6** (Poisson Summation Formula)**.** *If $f \in \mathfrak{F}$ then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

A proof of this theorem, as well as the convergence of (3.15) for functions in $\mathfrak{F}$ can be found in [20, Ch. 4].

The following is an example which will serve to familiarize the Fourier transform. This particular example will also be used later when proving Kronecker's limit formula.

Let $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{C}$ with $b := b_1 + b_2 i$. Let $f(z) = e^{-\pi a(z+b)^2}$. Before attempting to compute $\hat{f}$ we must verify that $f$ satisfies the properties (a) and (b) above. Since $f$ is an entire function, it is clear that (a) holds for any $\alpha > 0$. To prove (b), fix such an $\alpha$ and assume that $z = x + iy$ with $|y| < \alpha$. An explicit calculation of $|f(x+iy)|$ then shows that

$$\begin{aligned} |f(x+iy)| &= |e^{\pi a(y+b_2)^2} e^{\pi a(x+b_1)^2}| \\ &\leq Ce^{-\pi a(x+b_1)^2} \end{aligned}$$

where $C > 0$ is a constant. Thus, we would like to find $\beta > 0$ such that

$$Ce^{-\pi a(x+b_1)^2} < \frac{\beta}{1+x^2}.$$

Rearranging this inequality shows that it suffices to find a $\beta > 0$ such that

$$1 + x^2 < \frac{\beta}{C} e^{\pi a(x+b_1)^2}$$

Further, note that the series expansion of the exponential shows

$$1+\pi a(x+b_1)^2 < e^{\pi a(x+b_1)^2},$$

so in fact it suffices to find $\beta > 0$ such that

$$1+x^2 < \frac{\beta}{C}\left(1+\pi a(x+b_1)^2\right).$$

This is equivalent to

$$\beta > \frac{C(1+x^2)}{1+\pi a(x+b_1)^2}.$$

The right side of this inequality is bounded, so such a $\beta$ exists and $f$ satisfies property (b). We may now calculate the Fourier transform of $f$ with the following integral

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi a(x+b)^2} e^{-2\pi ixy} dx. \tag{3.16}$$

This can be computed by transforming it into a Gaussian integral. First note that

$$\begin{aligned} e^{-\pi a(x+b)^2} e^{-2\pi ixy} &= e^{-\pi a(x^2+2bx+2ixy/a+b^2)} \\ &= e^{-\pi ab^2} e^{-\pi a(x^2+c_y x)} \end{aligned}$$

where $c_y := 2(b+\frac{iy}{a})$. Completing the square on $x^2+c_y x$ in the above expression shows

$$\begin{aligned} e^{-\pi a(x+b)^2} e^{-2\pi ixy} &= e^{-\pi ab^2} e^{\pi a c_y^2/4} e^{-\pi a(x+c_y/2)^2} \\ &= e^{-\pi a(b^2-c_y^2/4)} e^{-\pi a(x+c_y/2)^2}. \end{aligned} \tag{3.17}$$

Thus, computing the integral in (3.16) amounts to computing a Gaussian integral of the form

$$I := \int_{-\infty}^{\infty} e^{-\pi a(x+\xi)^2} dx$$

where $a \in \mathbb{R}_{>0}$ and $\xi \in \mathbb{C}$. Write $\xi = r+is$ and define the function $\tilde{f}(z) = e^{-\pi a(z+r)^2}$ on the complex variable $z$. We will compute this Gaussian by integrating $\tilde{f}(z)$ over an appropriate rectangular contour, $\gamma$. Let $R \in \mathbb{R}_{>0}$ and define $\gamma$ as the rectangular contour beginning with the segment from $-R$ to $R$, followed by the segment $R$ to

$R+is$, followed by the segment from $R+is$ to $-R+is$, and finished with the segment from $-R+is$ to $-R$.

Using Cauchy's theorem and splitting the integral over $\gamma$ into integrals over each of the segments, we find

$$0 = \int_\gamma \tilde{f}(z)dz = I_1 + I_2 + I_3 + I_4$$

where the integrals $I_1, I_2, I_3$ and $I_4$ are defined by

$$\begin{aligned} I_1 &= \int_{-R}^{R} e^{-\pi a(x+r)^2} dx \\ I_2 &= i\int_0^s e^{-\pi a(R+iy+r)^2} dy \\ I_3 &= -\int_{-R}^{R} e^{-\pi a(x+\xi)^2} dx \\ I_4 &= -i\int_0^s e^{-\pi a(-R+iy+r)^2} dy. \end{aligned}$$

We then note that there exist constants $C_2$ and $C_4$ such that $|I_2| \leq C_2 e^{-\pi a(r+R)^2}$ and $|I_4| \leq C_4 e^{-\pi a(r-R)^2}$. Thus, as $|R| \to \infty$ we have that $I_2 \to 0$ and $I_4 \to 0$. This implies that

$$\int_{-\infty}^{\infty} e^{-\pi a(x+\xi)^2} dx = \int_{-\infty}^{\infty} e^{-\pi a(x+r)^2} dx = \sqrt{\frac{\pi}{\pi a}} = \frac{1}{\sqrt{a}}$$

where the last equalities come from the well known expression for the Gaussian integral. Using this along with (3.17) the integral in (3.16) becomes

$$\hat{f}(y) = \frac{e^{-\pi a(b^2 - c_y^2/4)}}{\sqrt{a}}.$$

But $c_y = 2(b + \frac{iy}{a})$ so $b^2 - \frac{c_y^2}{4} = \frac{y^2}{a^2} - \frac{2biy}{a}$ meaning that the Fourier transform of $f(z)$ is given by

$$\hat{f}(y) = \frac{e^{-\pi(y^2/a - 2biy)}}{\sqrt{a}}. \tag{3.18}$$

## 3.6 A Proof of the Limit Formula

We now return to the proof of Theorem 3.1. This section uses the same notation introduced in Section 3.1. Begin by introducing a new function related to the Siegel function $g_{t_1,t_2}(\tau)$. Let $\tau \in \mathbb{H}$ and define $f_\tau : \mathbb{C} \to \mathbb{C}$ by

$$f_\tau(z) = q_\tau^{1/12}(q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}).$$

Letting $z = u - v\tau$ we have that

$$\begin{aligned}
g_{-v,u}(\tau) &= -q_\tau^{\frac{1}{2}(v^2+v+1/6)} e^{-2\pi i u(v+1)/2}(1 - q_z) \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}) \\
&= -q_z^{1/2} q_\tau^{v/2} e^{-2\pi i u(v+1)/2} q_\tau^{v^2/2} q_\tau^{1/12} q_z^{-1/2}(1 - q_z) \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}) \\
&= q_z^{1/2} q_\tau^{v/2} e^{-2\pi i u(v+1)/2} q_\tau^{v^2/2} q_\tau^{1/12}(q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}) \\
&= q_z^{1/2} q_\tau^{v/2} e^{-2\pi i u(v+1)/2} q_\tau^{v^2/2} f_\tau(u - v\tau).
\end{aligned}$$

Now, consider the expression

$$q_z^{1/2} q_\tau^{v/2} = e^{\pi i z}(e^{\pi i \tau})^v.$$

Since $v \in \mathbb{R}$ it is not necessarily the case that $(e^{\pi i \tau})^v = e^{\pi i \tau v}$. However, it is true that $|(e^{\pi i \tau})^v| = |e^{\pi i \tau v}|$. This is because in general, given $r \in \mathbb{R}$ and $w \in \mathbb{C}$ we have

$$\begin{aligned}
|(e^w)^r| &= \left|e^{r\mathrm{Log}(e^w)}\right| \\
&= \left|e^{r\log(e^{\mathrm{Re}(w)})}\right| \\
&= \left|e^{r\mathrm{Re}(w)}\right| \\
&= \left|e^{rw}\right|.
\end{aligned} \tag{3.19}$$

Thus

$$\begin{aligned} |q_z^{1/2} q_\tau^{v/2}| &= \left|e^{\pi i z} e^{\pi i v \tau}\right| \\ &= \left|e^{\pi i (u - v\tau)} e^{\pi i v \tau}\right| \\ &= 1. \end{aligned}$$

This shows that

$$|g_{-v,u}(\tau)| = \left| f_\tau(u - v\tau) q_\tau^{v^2/2} \right|,$$

and thus, to verify (3.2) it suffices to prove $E_{u,v}(\tau, 1) = -2\pi \log \left| f_\tau(u - v\tau) q_\tau^{v^2/2} \right|$.

A main idea in the proof is to notice that it is enough to prove Kronecker's limit formula for $u, v \in [0, 1)$, because of the periodicity of the expressions involved. The bulk of the argument for this is given in the following lemma.

**Lemma 3.7.** *If $u, v \in \mathbb{R}$ and $\tau \in \mathbb{H}$ then the function $F_\tau(u - v\tau) := \left| f_\tau(u - v\tau) q_\tau^{v^2/2} \right|$ is periodic in $u$ and $v$ with period 1.*

*Proof.* For the remainder of this proof, let $z = u - v\tau$. Then $z + 1 = (u + 1) - v\tau$ so to prove periodicity in $u$ we need to show that $f_\tau(z + 1) = f_\tau(z)$. This follows immediately by noticing that

$$q_{z+1} = e^{2\pi i} q_z = q_z.$$

Thus $f_\tau(z + 1) = f_\tau(z)$ and in turn $F_\tau(z + 1) = F_\tau(z)$.

For periodicity in $v$ note that $z - \tau = u - (v + 1)\tau$, so we need to prove that $F_\tau(z - \tau) = F_\tau(z)$. Begin by noticing that

$$q_{z-\tau} = e^{-2\pi i \tau} q_z = q_z q_\tau^{-1}.$$

Applying this relationship to the definition of $f_\tau(z)$ it follows that

$$\begin{aligned} f_\tau(z-\tau) &= q_\tau^{1/12}\Big[(q_z q_\tau^{-1})^{1/2} - (q_z q_\tau^{-1})^{-1/2}\Big]\prod_{n=1}^{\infty}(1-q_\tau^{n-1}q_z)(1-q_\tau^{n+1}q_z^{-1}) \\ &= q_\tau^{1/12}(q_z q_\tau^{-1})^{1/2}(1-q_\tau q_z^{-1})\prod_{n=1}^{\infty}(1-q_\tau^{n-1}q_z)(1-q_\tau^{n+1}q_z^{-1}) \\ &= q_\tau^{1/12}(q_z q_\tau^{-1})^{1/2}(1-q_\tau q_z^{-1})P_\tau(z) \end{aligned} \tag{3.20}$$

where the function $P_\tau(z) = \prod_{n=1}^{\infty}(1-q_\tau^{n-1}q_z)(1-q_\tau^{n+1}q_z^{-1})$ has been introduced for ease of notation. The factors of $P_\tau(z)$ may be rearranged because the product converges absolutely, which allows $P_\tau(z)$ to be rewritten as two infinite products and reindexed as follows:

$$\begin{aligned} P_\tau(z) &= \left(\prod_{n=1}^{\infty}(1-q_\tau^{n-1}q_z)\right)\left(\prod_{n=1}^{\infty}(1-q_\tau^{n+1}q_z^{-1})\right) \\ &= (1-q_z)(1-q_\tau q_z)\left(\prod_{n=3}^{\infty}(1-q_\tau^{n-1}q_z)\right)\left(\prod_{n=1}^{\infty}(1-q_\tau^{n+1}q_z^{-1})\right) \\ &= (1-q_z)(1-q_\tau q_z)\left(\prod_{n=2}^{\infty}(1-q_\tau^{n}q_z)\right)\left(\prod_{n=2}^{\infty}(1-q_\tau^{n}q_z^{-1})\right) \\ &= (1-q_z)(1-q_\tau q_z)\prod_{n=2}^{\infty}(1-q_\tau^{n}q_z)(1-q_\tau^{n}q_z^{-1}) \\ &= (1-q_z)(1-q_\tau q_z^{-1})^{-1}\prod_{n=1}^{\infty}(1-q_\tau^{n}q_z)(1-q_\tau^{n}q_z^{-1}). \end{aligned}$$

Combining this with the expression for $f_\tau(z-\tau)$ in (3.20) we find that

$$\begin{aligned} f_\tau(z-\tau) &= q_\tau^{1/12}(q_z q_\tau^{-1})^{1/2}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^{n}q_z)(1-q_\tau^{n}q_z^{-1}) \\ &= q_\tau^{1/12}(q_z q_\tau^{-1})^{1/2}q_z^{1/2}(q_z^{-1/2}-q_z^{1/2})\prod_{n=1}^{\infty}(1-q_\tau^{n}q_z)(1-q_\tau^{n}q_z^{-1}) \\ &= -q_z q_\tau^{-1/2}f_\tau(z). \end{aligned} \tag{3.21}$$

Finally, we make note that $q_z = e^{2\pi i(u-v\tau)} = q_u q_{v\tau}^{-1}$ and that

$$|q_{v\tau}^{-1}q_\tau^{v}| = 1. \tag{3.22}$$

If $q_\tau \in \mathbb{R}$ this identity is obvious because $q_\tau^v = q_{v\tau}$. In general, though, when $q_\tau \notin \mathbb{R}$ it is not necessarily true that $(q_\tau)^v = q_{\tau v}$. However in this case we may use (3.19) to see that $|(q_\tau)^v| = |q_{v\tau}|$, so (3.22) still holds. Using this along with (3.21) we find

$$\begin{aligned} F_\tau(z-\tau) &= \left| f_\tau(z-\tau) q_\tau^{(v+1)^2/2} \right| \\ &= \left| q_z q_\tau^{-1/2} f_\tau(z) q_\tau^{v^2/2} q_\tau^v q_\tau^{1/2} \right| \\ &= \left| q_u q_{v\tau}^{-1} q_\tau^{-1/2} f_\tau(z) q_\tau^{v^2/2} q_\tau^v q_\tau^{1/2} \right| \\ &= \left| f_\tau(z) q_\tau^{v^2/2} \right| \\ &= F_\tau(z). \end{aligned}$$

This proves that $F_\tau(z)$ is periodic in $v$, which completes the proof of the lemma. □

It is now not a difficult task to note that $E_{u,v}(\tau, s)$ is also periodic in $u$ and $v$ with period 1. This is immediately clear by replacing $u$ and $v$ by $u+1$ and $v+1$, respectively, in (3.1). Moreover, Lemma 3.7 shows that the right hand side of Kronecker's limit formula is also periodic in $u$ and $v$ with period 1. Thus, it suffices to prove Kronecker's limit formula for $u, v \in [0, 1)$ and for the remainder of the proof it is assumed that $u$ and $v$ are in this interval.

We now return to the definition of $E_{u,v}(\tau, s)$ given in (3.1). By separating the terms with $m = 0$ from the rest of the sum the expression becomes

$$\begin{aligned} E_{u,v}(\tau, s) &= y^s {\sum_{n\in\mathbb{Z}}}' \frac{e^{2\pi i n v}}{n^{2s}} + y^s {\sum_{m\in\mathbb{Z}}}' e^{2\pi i m u} \sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n v}}{|m\tau + n|^{2s}} \\ &= y^s \big[ S_1(s) + S_2(\tau, s) \big] \end{aligned} \tag{3.23}$$

where the functions $S_1(s)$ and $S_2(\tau, s)$ are defined for convenience as the first and second summations, respectively, in the expression for $E_{u,v}(\tau, s)$. When $s = 1$ the sum $S_1(s)$ can be recognized as a standard Fourier series as follows. By summing the Fourier series given in Section 3.3 we find that the Fourier series of the function $f(x) = x^2 - x + \frac{1}{6}$ defined on $[0, 1]$ is given by

$$S_f(x) = {\sum_{n\in\mathbb{Z}}}' \frac{1}{2\pi^2 n^2} e^{2\pi i n x}.$$

Of course, convergence of this series must be discussed. First note that since $f$ and $f'$ are continuous it is immediate from Theorem 3.5 that $S_f(x)$ converges to $f(x)$ on the interval $(0,1)$. Moreover, because $f(0) = f(1)$ it follows that $f_{\text{per}}(0+) = f_{\text{per}}(0-) = f(0)$ and $f_{\text{per}}(1+) = f_{\text{per}}(1-) = f(1)$ where $f_{\text{per}}(x)$ is the periodic extension of $f$ as defined in (3.8). Thus, Theorem 3.5 also implies that $S_f(x)$ converges to $f(x)$ for $x = 0$ and $x = 1$.

This discussion shows that the expression $S_1(1)$ is given by

$$S_1(1) = \sum_{n\in\mathbb{Z}}{}' \frac{e^{2\pi inv}}{n^2} = 2\pi^2\left(v^2 - v + \frac{1}{6}\right) \tag{3.24}$$

for all $v \in [0,1]$.

After a considerable amount of algebraic manipulation the function $S_2(\tau, s)$ can be written in terms of the function $K_{s-\frac{1}{2}}(a,b)$ for suitable choices of $a$ and $b$. Thus, when $s = 1$ we may use the expression for $K_{\frac{1}{2}}(a,b)$ given in (3.14) to greatly simplify this term. We now embark on this task. From (3.9) using $a = \pi|m\tau + n|^2$ it follows that

$$\frac{1}{|m\tau+n|^{2s}} = \frac{\pi^s}{\Gamma(s)}\int_0^\infty e^{-\pi t|m\tau+n|^2}t^{s-1}dt.$$

Using this fact we have

$$S_2(\tau, s) = \frac{\pi^s}{\Gamma(s)}\sum_{m\in\mathbb{Z}}{}' e^{2\pi imu}\sum_{n\in\mathbb{Z}} e^{2\pi inv}\int_0^\infty e^{-\pi t|m\tau+n|^2}t^{s-1}dt. \tag{3.25}$$

Now, denoting $\tau = x + iy$, the expression $|m\tau + n|^2$ can be rewritten as

$$|m\tau + n|^2 = |mx + n + imy|^2 = (mx+n)^2 + m^2y^2.$$

We now substitute this expression for $|m\tau + n|^2$ into (3.25) and then combine the exponentials involving $n$.

$$\begin{aligned} S_2(\tau, s) &= \frac{\pi^s}{\Gamma(s)}\sum_{m\in\mathbb{Z}}{}' e^{2\pi imu}\sum_{n\in\mathbb{Z}} e^{2\pi inv}\int_0^\infty e^{-\pi t(n+mx)^2}e^{-\pi tm^2y^2}t^{s-1}dt \\ &= \frac{\pi^s}{\Gamma(s)}\sum_{m\in\mathbb{Z}}{}' e^{2\pi imu}\int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi t[(n+mx)^2-2inv/t]}e^{-\pi tm^2y^2}t^{s-1}dt \end{aligned}$$

Finally, note that the exponent involving $n$ can be rewritten by realizing it as a quadratic in $n + mx$ and then completing the square as follows:

$$\begin{aligned} -\pi t \left[ (n+mx)^2 - \frac{2inv}{t} \right] &= -\pi t \left[ (n+mx)^2 - \frac{2iv}{t}(n+mx) \right] - 2\pi imvx \\ &= -\pi t \left( n + mx - \frac{iv}{t} \right)^2 - 2\pi imvx - \frac{\pi v^2}{t}. \end{aligned}$$

Using this expression for the exponent it follows that

$$S_2(\tau, s) = \frac{\pi^s}{\Gamma(s)} {\sum_{m\in\mathbb{Z}}}' e^{2\pi im(u-vx)} \int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi t(n+mx-iv/t)^2} e^{-\pi(tm^2y^2+v^2/t)} t^{s-1} dt.$$

The integral in this expression is beginning to take a similar form to the integral $K_s(a, b)$ introduced in (3.10), but the $(n + mx - iv/t)^2$ factor in the exponent makes it difficult to put it in the exact form needed. To remedy this, we will take the Fourier transform of $e^{-\pi t(n+mx-iv/t)^2}$ and then use the Poisson summation formula from (3.6) to rewrite $\sum_{n\in\mathbb{Z}} e^{-\pi t(n+mx-iv/t)^2}$ as a sum over the Fourier transforms.

Note, however that $e^{-\pi t(n+mx-iv/t)^2}$ is an expression of the form $e^{-\pi a(n+b)^2}$ with $a = t$ and $b = mx - \frac{iv}{t}$. The Fourier transform of this expression was given in (3.18). Using this result along with Theorem 3.6 it follows that if $f(n) = e^{-\pi t(n+mx-iv/t)^2}$ then

$$\begin{aligned} \sum_{n\in\mathbb{Z}} f(n) &= \sum_{n\in\mathbb{Z}} \hat{f}(n) \\ &= \sum_{n\in\mathbb{Z}} \frac{e^{-\pi(n^2/t-2in(mx-iv/t))}}{\sqrt{t}} \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{t}} e^{-\pi n^2/t} e^{2\pi in(mx-iv/t)}. \end{aligned}$$

Thus, at last an expression for $S_2(\tau, s)$ emerges:

$$\begin{aligned} S_2(\tau, s) &= \frac{\pi^s}{\Gamma(s)} {\sum_{m\in\mathbb{Z}}}' e^{2\pi im(u-vx)} \int_0^\infty \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{t}} e^{-\pi n^2/t} e^{2\pi in(mx-iv/t)} e^{-\pi(tm^2y^2+v^2/t)} t^{s-1} dt \\ &= \frac{\pi^s}{\Gamma(s)} {\sum_{m\in\mathbb{Z}}}' e^{2\pi im(u-vx)} \sum_{n\in\mathbb{Z}} e^{2\pi inmx} \int_0^\infty e^{-\pi[tm^2y^2+(n-v)^2/t]} t^{s-\frac{3}{2}} dt \\ &= \frac{\pi^s}{\Gamma(s)} {\sum_{m\in\mathbb{Z}}}' e^{2\pi im(u-vx)} \sum_{n\in\mathbb{Z}} e^{2\pi inmx} K_{s-\frac{1}{2}}(my\sqrt{\pi}, (n-v)\sqrt{\pi}). \end{aligned}$$

Letting $s = 1$ and using (3.14) this becomes

$$\begin{aligned} S_2(\tau, 1) &= \pi \sum_{m\in\mathbb{Z}}{}' e^{2\pi i m(u-vx)} \sum_{n\in\mathbb{Z}} e^{2\pi i n m x} K_{\frac{1}{2}}(my\sqrt{\pi}, (n-v)\sqrt{\pi}) \\ &= \pi \sum_{m\in\mathbb{Z}}{}' e^{2\pi i m(u-vx)} \sum_{n\in\mathbb{Z}} e^{2\pi i n m x} \cdot \frac{\sqrt{\pi}}{|ym\sqrt{\pi}|} e^{-2|\pi y m(n-v)|} \\ &= \pi \sum_{m\in\mathbb{Z}}{}' e^{2\pi i m(u-vx)} \sum_{n\in\mathbb{Z}} e^{2\pi i n m x} \cdot \frac{1}{y|m|} e^{-2\pi y|m(n-v)|}. \end{aligned} \tag{3.26}$$

where the last equality follows since $y > 0$ by assumption. The goal now will be to show that the summations in the above expression can be interchanged. At this point, we must split into two cases: one where $v \neq 0$ and one where $v = 0$. The differences between the two cases are subtle, and the arguments are almost identical. First, recall the power series formula

$$-\text{Log}(1-r) = \sum_{n=1}^{\infty} \frac{r^n}{n} \qquad \text{if } |r| < 1. \tag{3.27}$$

In fact, this formula still holds when $|r| = 1$ provided that $r \neq 1$.

Consider the case when $v \neq 0$. Then, the order of summation in (3.26) may be switched. To see why, recall that given a sequence $(a_{mn})$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}$$

provided that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|$ converges (Theorem 1 in [24, Ch. 16.3]). This property, is met by the double series (3.26) since

$$\begin{aligned} &\frac{\pi}{y} \sum_{m\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}} \left| e^{2\pi i m(u-vx)} e^{2\pi i n m x} \cdot \frac{1}{|m|} e^{-2\pi y|m(n-v)|} \right| \\ &= \frac{\pi}{y} \sum_{m\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}} \frac{1}{|m|} e^{-2\pi y|m(n-v)|}, \end{aligned}$$

which can be shown to converge when $v \neq 0$. Thus, in (3.26) the summations may be interchanged in this case to obtain

$$S_2(\tau, 1) = \frac{\pi}{y} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{|m|} e^{2\pi i[m(u-vx)+nmx+iy|m(n-v)|]}. \tag{3.28}$$

In this expression, the series expansion for $-\text{Log}(1-r)$ appears with

$$r = e^{2\pi i[\pm(u-vx)\pm nx+iy|n-v|]}.$$

Moreover, we have

$$\left|e^{2\pi i[\pm(u-vx)\pm nx+iy|n-v|]}\right| = e^{-2\pi y|n-v|} < 1.$$

We will now consider four cases to deal with the double summation. Keep in mind that $v \in (0,1)$ so if $n > 0$ then $n > v$, which means $|n-v| = n-v$. Otherwise, if $n \leq 0$ then $|n-v| = -(n-v)$. Also, recall that $\tau = x+iy$ with $y > 0$ and $z = u - v\tau$.
**Case 1:** If $n, m > 0$ then the summation is given by

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i[u-vx+nx+iy(n-v)]m} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i(z+n\tau)m} \\ &= \sum_{n=1}^{\infty} -\text{Log}(1-q_z q_\tau^n).\end{aligned}$$

**Case 2:** If $n > 0$ and $m < 0$ then the summation is given by

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i[-(u-vx)-nx+iy(n-v)]m} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i(-\overline{z}-n\overline{\tau})m} \\ &= \sum_{n=1}^{\infty} -\text{Log}(1-q_{\overline{z}}^{-1} q_{\overline{\tau}}^{-n}).\end{aligned}$$

**Case 3:** If $n \leq 0$ and $m > 0$ then the summation is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i[u-vx-nx+iy(n+v)]m} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i(\overline{z}-n\overline{\tau})m} \\ &= \sum_{n=0}^{\infty} -\text{Log}(1-q_{\overline{z}} q_{\overline{\tau}}^{-n}).\end{aligned}$$

**Case 4:** If $n \leq 0$ and $m < 0$ the summation is given by

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i[-(u-vx)+nx+iy(n+v)]m} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i(-z+n\tau)m}$$
$$= \sum_{n=0}^{\infty} -\mathrm{Log}(1-q_z^{-1}q_\tau^n).$$

Note that given $w \in \mathbb{C}$ we have $\overline{q_w} = \overline{e^{2\pi i w}} = e^{-2\pi i \overline{w}} = q_{\overline{w}}^{-1}$. Using this along with the simplifications in each of the above cases, we can rewrite the double summation. The $n = 0$ terms are given by

$$\begin{aligned}
-\mathrm{Log}(1-q_{\overline{z}}) - \mathrm{Log}(1-q_z^{-1}) &= -\mathrm{Log}\left[(1-q_{\overline{z}})(1-q_z^{-1})\right] \\
&= -\mathrm{Log}\left[q_{\overline{z}}^{1/2}(q_{\overline{z}}^{-1/2}-q_{\overline{z}}^{1/2})\cdot q_z^{-1/2}(q_z^{1/2}-q_z^{-1/2})\right] \\
&= -\log\left[\left|q_z^{-1/2}\right|^2\left|q_z^{1/2}-q_z^{-1/2}\right|^2\right] \\
&= -\log\left[e^{-2\pi v y}\left|q_z^{1/2}-q_z^{-1/2}\right|^2\right] \\
&= 2\pi v y - 2\log\left|q_z^{1/2}-q_z^{-1/2}\right|.
\end{aligned} \tag{3.29}$$

Note that for $w, z \in \mathbb{C}$ with $wz \in \mathbb{R}$ the use of the identity $\mathrm{Log}(wz) = \mathrm{Log}(w)+\mathrm{Log}(z)$ as above is justified. To see why, let $w = |w|e^{i\theta_1}$ and $z = |z|e^{i\theta_2}$ with $\theta_1, \theta_2 \in (-\pi, \pi)$. Then since $wz \in \mathbb{R}$ we have that $\theta_1 + \theta_2 = 2\pi n$ for some $n \in \mathbb{Z}$. However, since $\theta_1, \theta_2 \in (-\pi, \pi)$ it must be that $\theta_1 + \theta_2 = 0$. Thus

$$\begin{aligned}
\mathrm{Log}(w) + \mathrm{Log}(z) &= \log|w| + i\theta_1 + \log|z| + i\theta_2 \\
&= \log|wz| + i(\theta_1+\theta_2) \\
&= \mathrm{Log}(wz)
\end{aligned}$$

This also applies for the rest of the terms of the double summation in $E_{u,v}(\tau, 1)$ which are given by

$$
\begin{aligned}
-\sum_{n=1}^{\infty} &\left[\operatorname{Log}(1-q_z q_\tau^n)+\operatorname{Log}(1-q_{\bar{z}}^{-1} q_{\bar{\tau}}^{-n})+\operatorname{Log}(1-q_{\bar{z}} q_{\bar{\tau}}^{-n})+\operatorname{Log}(1-q_z^{-1} q_\tau^n)\right] \\
&= -\sum_{n=1}^{\infty} \operatorname{Log}[(1-q_z q_\tau^n)(1-q_{\bar{z}}^{-1} q_{\bar{\tau}}^{-n})(1-q_{\bar{z}} q_{\bar{\tau}}^{-n})(1-q_z^{-1} q_\tau^n)] \\
&= -\sum_{n=1}^{\infty} \operatorname{Log}[(1-q_z q_\tau^n)(\overline{1-q_z q_\tau^n})(1-q_z^{-1} q_\tau^n)(\overline{1-q_z^{-1} q_\tau^n})] \\
&= -2\sum_{n=1}^{\infty} \log\left|(1-q_z q_\tau^n)(1-q_z^{-1} q_\tau^n)\right|.
\end{aligned}
$$

Thus, it has been shown that when $v \neq 0$

$$
\begin{aligned}
E_{u,v}(\tau,1) &= 2\pi^2\left(v^2+\frac{1}{6}\right)y - 2\pi\log\left|q_z^{1/2}-q_z^{-1/2}\right| - 2\pi\sum_{n=0}^{\infty}\log\left|(1-q_z q_\tau^n)(1-q_z^{-1} q_\tau^n)\right| \\
&= 2\pi^2\left(v^2+\frac{1}{6}\right)y - 2\pi\log\left|(q_z^{1/2}-q_z^{-1/2})\prod_{n=1}^{\infty}(1-q_z q_\tau^n)(1-q_z^{-1} q_\tau^n)\right|.
\end{aligned}
$$

This expression also holds when $v = 0$, which we now discuss. Assume $v = 0$ and begin by separating the $n = 0$ terms from (3.26). We obtain

$$
\begin{aligned}
\frac{y}{\pi}\cdot S_2(\tau,1) &= \sum_{m\in\mathbb{Z}}{}' \frac{e^{2\pi i m u}}{|m|} + \sum_{m\in\mathbb{Z}}{}' e^{2\pi i m(u-vx)} \sum_{n\in\mathbb{Z}}{}' e^{2\pi i n m x}\cdot\frac{1}{|m|}e^{-2\pi y|m(n-v)|} \\
&= \sum_{m\in\mathbb{Z}}{}' \frac{e^{2\pi i m u}}{|m|} + \sum_{n\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' e^{2\pi i m(u-vx)} e^{2\pi i n m x}\cdot\frac{1}{|m|}e^{-2\pi y|m(n-v)|}.
\end{aligned}
$$

The exchange of the inner summations follows from the same argument used in the case where $v \neq 0$, because now the $n = 0$ terms have been separated before attempting to swap the order of summation. Moreover, the series

$$
\sum_{m\in\mathbb{Z}}{}' \frac{e^{2\pi i m u}}{|m|}
$$

converges by the note under (3.27) since $u \neq 0$ (because $v = 0$ and we require that $u$ and $v$ are not both integers). Note that this series may not converge in the case when $v \neq 0$ because then the exponential is of the form $e^{2\pi i m(u-vx)}$ and it is possible that $u - vx \in \mathbb{Z}$ for some $x$. This is why this case needs to be dealt with separately.

We have,

$$S_2(\tau,1) = \frac{\pi}{y}\sum_{m\in\mathbb{Z}}{}' \frac{e^{2\pi imu}}{|m|} + \frac{\pi}{y}\sum_{n\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' \frac{1}{|m|} e^{2\pi i[m(u-vx)+nmx+iy|m(n-v)|]}, \tag{3.30}$$

for $v = 0$. Note that the double summations in (3.28) and (3.30) differ only by the $n = 0$ terms. The expressions for the double summations in each of the four cases discussed above are still valid, except that the $n = 0$ terms in case 3 and case 4 are excluded. However, we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}{}' \frac{e^{2\pi imu}}{|m|} &= \sum_{m=1}^{\infty} \frac{e^{2\pi imu}}{m} + \sum_{m=1}^{\infty} \frac{e^{-2\pi imu}}{m} \\
&= -\mathrm{Log}(1-q_u) - \mathrm{Log}(1-q_u^{-1}) \\
&= -\mathrm{Log}(1-q_{\overline{z}}) - \mathrm{Log}(1-q_z^{-1}),
\end{aligned}$$

where the last line follows since $v = 0$ so $z = \overline{z} = u$. This expression is the same as the $n = 0$ terms in (3.29). Thus (3.28) and (3.30) are equal.

There is now only one small algebraic manipulation remaining to put this expression in the form given in Kronecker's limit formula. We make the observation that

$$\begin{aligned}
-2\pi \log \left| q_\tau^{1/12} q_\tau^{v^2/2} \right| &= -\pi \log \left| e^{\pi i\tau/3} e^{2\pi iv^2\tau} \right| \\
&= -\pi \log \left| e^{\pi i(x+iy)/3 + 2\pi iv^2(x+iy)} \right| \\
&= -\pi \log \left( e^{-\pi y/3 - 2\pi v^2 y} \right) \\
&= \frac{\pi^2 y}{3} + 2\pi^2 v^2 y \\
&= 2\pi^2 \left( v^2 + \frac{1}{6} \right) y.
\end{aligned}$$

Then, substituting the expression $-2\pi\log\left|q_\tau^{1/12}q_\tau^{v^2/2}\right|$ for $2\pi^2\left(v^2+\frac{1}{6}\right)y$ in the right side of the expression for $E_{u,v}(\tau,1)$ results in Kronecker's limit formula.

## 3.7 An Expression for $L'_{\mathfrak{m}}(0,\psi)$

Assume that $k$ is a quadratic imaginary number field, $\mathfrak{m} \neq (1)$ is an integral ideal of $k$, and and that $\psi$ is a proper character of $Cl_{\mathfrak{m}}(k)$. Using the results proven up to this point, it is now straightforward to produce a useful expression for $L'_{\mathfrak{m}}(0,\psi)$. Here, we assume familiarity with the notation of Corollary 1.23, Theorem 3.1 (Kronecker's limit formula), and Theorem 3.4.

Firstly, using Theorem 3.4 along with Kronecker's limit formula produces

$$\begin{aligned} L_{\mathfrak{m}}(1,\overline{\psi}) &= \frac{2\overline{\psi}(\mathfrak{a})}{w_{\mathfrak{m}}\tau(\psi_f,y)\sqrt{|d_k|}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(\mathfrak{a}\mathfrak{b}_R) E_{u(\mathfrak{b}_R),v(\mathfrak{b}_R)}(\tau_R,1) \\ &= \frac{-4\pi\overline{\psi}(\mathfrak{a})}{w_{\mathfrak{m}}\tau(\psi_f,y)\sqrt{|d_k|}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(\mathfrak{a}\mathfrak{b}_R) \log\left|g_{-v(\mathfrak{b}_R),u(\mathfrak{b}_R)}(\tau_R)\right|. \end{aligned}$$

Using this expression along with Corollary 1.23, and writing $u := u(\mathfrak{b}_R)$ and $v := v(\mathfrak{b}_R)$ gives

$$\begin{aligned} L'_{\mathfrak{m}}(0,\psi) &= -\frac{2}{w_{\mathfrak{m}}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(\mathfrak{a}\mathfrak{b}_R) \log|g_{-v,u}(\tau_R)| \\ &= -\frac{1}{12N_{\mathfrak{m}}w_{\mathfrak{m}}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(\mathfrak{a}\mathfrak{b}_R) \log\left|g_{-v,u}(\tau_R)^{12N_{\mathfrak{m}}}\right|^2, \end{aligned} \tag{3.31}$$

where $N_{\mathfrak{m}}$ is the smallest positive integer in $\mathfrak{m}$. This expression will be used in Chapter 5 to provide verification of (2.1).

# 4 Elliptic Curves and Automorphic Forms

Prior to this chapter the Siegel functions $g_{t_t,t_2}(\tau)$ were introduced when discussing Kronecker's second limit formula (Theorem 3.1). For that theorem, we did not need any properties of the Siegel functions beyond the product expansion. Moving forward, it will be necessary to understand the Siegel functions more completely in order to explain how they generate Stark units when evaluated at CM points. To introduce the required properties of the Siegel functions requires some background knowledge from the theory of elliptic curves, elliptic functions, and automorphic forms, each of which is introduced here.

## 4.1 Elliptic Curves as Complex Tori

Given a basis $\{\omega_1, \omega_2\}$ for $\mathbb{C}$ over $\mathbb{R}$ the lattice generated by these elements is defined as

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} = \{a_1\omega_1 + a_2\omega_2 \mid a_1, a_2 \in \mathbb{Z}\}$$

A complex torus is any quotient of the form $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a lattice. Clearly, since $\Lambda$ is a subgroup of the abelian group $\mathbb{C}$, complex tori are abelian groups. In addition, every complex torus is a connected complex manifold of dimension 1. Specifically, every complex torus has an atlas of charts such that the transition maps are holomorphic. Lastly, the binary operation as well as the inverse operation on $\mathbb{C}/\Lambda$ are holomorphic as well. Putting all of this together means that $\mathbb{C}/\Lambda$ is a Riemann surface and a complex Lie group.

The first order of business is to discuss maps between complex tori. The maps of interest are holomorphic group homomorphisms. These maps can be described in a remarkably simple way as functions on $\mathbb{C}$. This fact is the content of the following proposition. For the proof we follow [5, Ch. 1.3].

**Theorem 4.1.** *(a) If $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is a holomorphic group homomorphism then there exists $\alpha \in \mathbb{C}$ with $\alpha\Lambda \subseteq \Lambda'$ such that the map $\varphi_\alpha : \mathbb{C} \to \mathbb{C}$ defined by $\varphi_\alpha(z) = \alpha z$ makes the diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\varphi_\alpha} & \mathbb{C} \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \xrightarrow{\varphi} & \mathbb{C}/\Lambda' \end{array}$$

*commute. Here $\pi$ and $\pi'$ are the canonical projection maps.*

*(b) Conversely, every $\alpha \in \mathbb{C}$ with $\alpha\Lambda \subseteq \Lambda'$ gives a well defined holomorphic group homomorphism $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ given by $\varphi(z + \Lambda) = \alpha z + \Lambda'$.*

*(c) $\varphi$ is an isomorphism if and only if $\alpha\Lambda = \Lambda'$*

*Proof.* (a) We will use a lifting theorem from the topological theory of covering maps to lift the map $\varphi$ to a continuous, holomorphic map $\rho$ on $\mathbb{C}$. From properties of the map $\rho$ we will see that the map $\varphi_\alpha$ as defined in the statement of the theorem does indeed make the diagram commute.

To produce the lift, first note that we are given a continuous map $\varphi \circ \pi : \mathbb{C} \to \mathbb{C}/\Lambda'$. Moreover, it can be shown $\pi'$ is surjective, continuous, and that for every element $z + \Lambda' \in \mathbb{C}/\Lambda'$ there exists an open neighborhood $U$ of $z + \Lambda'$ that is evenly covered by $\pi'$. This means that $\pi' : \mathbb{C} \to \mathbb{C}/\Lambda'$ is a covering map. This fact along with the fact that the fundamental group of $\mathbb{C}$ is trivial since it is simply connected may be used in conjunction with the lifting theorem in [13, Lem. 79.1] to prove that a continuous map $\rho$ which makes the diagram commute exists. To see that $\rho$ is holomorphic note that local coordinates on the tori are given by restricting the maps $\pi$ and $\pi'$ to suitable open subsets of $\mathbb{C}$. Because $\pi$ and $\varphi$ are holomorphic, and $\pi'^{-1}$ is locally holomorphic this implies that $\rho$ is also holomorphic.

It will now be shown that $\rho(z) + \Lambda' = \alpha z + \Lambda'$ for some $\alpha \in \mathbb{C}$ with $\alpha\Lambda \subseteq \Lambda'$. We will first show that the holomorphic function $\rho$ is $\Lambda$-periodic, thus bounded, thus constant by Liouville's theorem. This will imply that $\rho(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$.

Let $\lambda \in \Lambda$. Then using the commutative diagram we have

$$\rho(z + \lambda) + \Lambda' = \varphi(z + \lambda + \Lambda) = \varphi(z + \Lambda) = \rho(z) + \Lambda'.$$

Thus, for $\lambda \in \Lambda$ we have $f_\lambda(z) := \rho(z+\lambda) - \rho(z) \in \Lambda'$. So, the function $f_\lambda : \mathbb{C} \to \Lambda'$ is a continuous map into a discrete space, and since $\mathbb{C}$ is connected this implies $f_\lambda$ is constant. Indeed, if $z \in \mathbb{C}$ such that $f_\lambda(z) = w$ then $U_w := f_\lambda^{-1}(\{w\})$ is an open neighborhood of $z$ since $\{w\}$ is open in $\Lambda'$ and $f_\lambda$ is continuous. Moreover, for all $u \in U_w$ we have $f_\lambda(u) = w$. Thus, every $z \in \mathbb{C}$ has an open neighborhood on which $f_\lambda$ is constant. Since $\mathbb{C}$ is connected this implies that $f_\lambda$ is constant over all of $\mathbb{C}$.

Differentiating $f_\lambda(z) = \rho(z+\lambda) - \rho(z)$ then gives $\rho'(z+\lambda) = \rho'(z)$. Thus, $\rho'(z)$ is constant by Louiville's theorem and there exist $\alpha, \beta \in \mathbb{C}$ such that $\rho(z) = \alpha z + \beta$. Moreover, from the commutative diagram we have that

$$\alpha z + \beta + \Lambda' = \rho(z) + \Lambda' = \varphi(z + \Lambda).$$

In particular if $z = \lambda \in \Lambda$ it follows that

$$\alpha\lambda + \beta + \Lambda' = \rho(\lambda) + \Lambda' = \varphi(\lambda + \Lambda) = \Lambda'$$

since $\varphi$ is a homomorphism. And so

$$\beta + \Lambda' = \rho(0) + \Lambda' = \Lambda'.$$

These expressions together show that $\beta \in \Lambda'$ and so $\alpha\lambda \in \Lambda'$. Thus,

$$\varphi(z + \Lambda) = \rho(z) + \Lambda' = \alpha z + \Lambda'$$

with $\alpha\Lambda \subseteq \Lambda'$. This shows that in fact the map $\varphi_\alpha(z) = \alpha z$ does indeed make the diagram commute as required.

(b) Let $\alpha \in \mathbb{C}$ with $\alpha\Lambda \subseteq \Lambda'$ and define $\varphi(z+\Lambda) = \alpha z + \Lambda'$. This is well defined because if $w + \Lambda = z + \Lambda$ then $\alpha z - \alpha w = \alpha(z - w) \in \alpha\Lambda$ and since $\alpha\Lambda \subseteq \Lambda'$ we have that

$$\varphi(w + \Lambda) = \alpha w + \Lambda' = \alpha z + \Lambda' = \varphi(z + \Lambda).$$

Moreover, $\varphi$ is clearly a homomorphism and it is holomorphic because $\alpha z$ and the projection maps are holomorphic.

(c) First assume that $\alpha\Lambda = \Lambda'$. If this is the case $\alpha \neq 0$ so $\varphi_\alpha$ and $\pi'$ are surjective. Thus, $\varphi$ is also surjective. To see that $\varphi$ is also injective it suffices to show that $\ker(\varphi)$

is trivial. Let $w + \Lambda \in \ker(\varphi)$. Then

$$\varphi(w + \Lambda) = \alpha w + \Lambda' = \Lambda'$$

so $\alpha w \in \Lambda'$ and since $\Lambda' \subseteq \alpha\Lambda$ this implies $\alpha w \in \alpha\Lambda$. Thus $w + \Lambda = \Lambda$ and $\ker(\varphi)$ is trivial. This shows that $\alpha\Lambda = \Lambda'$ implies that $\varphi$ is bijective, and thus an isomorphism.

Conversely, assume that $\varphi$ is an isomorphism. We already know from (a) that $\alpha\Lambda \subseteq \Lambda'$. It remains to show that $\Lambda' \subseteq \alpha\Lambda$. Let $\lambda' \in \Lambda$. Then since $\alpha \neq 0$

$$\varphi\left(\frac{\lambda'}{\alpha} + \Lambda\right) = \lambda' + \Lambda' = \Lambda'.$$

Thus, $\lambda'/\alpha + \Lambda \in \ker(\varphi)$ and $\varphi$ is an injective homomorphism so $\lambda'/\alpha \in \Lambda$ which implies $\lambda' \in \alpha\Lambda$. This shows that $\varphi$ being an isomorphism implies $\alpha\Lambda = \Lambda'$. □

**Corollary 4.2.** *There exists a nonzero holomorphic homomorphism between* $\mathbb{C}/\Lambda$ *and* $\mathbb{C}/\Lambda'$ *if and only if there exists a nonzero* $\alpha \in \mathbb{C}$ *such that* $\alpha\Lambda \subseteq \Lambda'$. *Such a map is an isomorphism if and only if* $\alpha\Lambda = \Lambda'$.

Nonzero holomorphic homomorphisms between complex tori are often called isogenies, and isogenies from a torus to itself are called endomorphisms. Given a complex torus $\mathbb{C}/\Lambda$ the endomorphisms of $\mathbb{C}/\Lambda$ along with the zero map form a ring with the operations of function addition and function composition. This ring is denoted by $\text{End}(\mathbb{C}/\Lambda)$. Note that if $m \in \mathbb{Z}$. Then $m\Lambda \subset \Lambda$ so $m$ defines an endomorphism $[m] : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ given by $z + \Lambda \mapsto mz + \Lambda$. If these are the only endomorphisms on $\mathbb{C}/\Lambda$ then $\text{End}(\mathbb{C}/\Lambda) \cong \mathbb{Z}$. As the next proposition shows, if $\mathbb{C}/\Lambda$ has more endomorphisms than these, they have a very specific form.

**Theorem 4.3.** *The endomorphism ring of a complex torus* $\mathbb{C}/\Lambda$ *is isomorphic either to* $\mathbb{Z}$ *or to an order in a quadratic imaginary number field.*

*Proof.* Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ be a lattice in $\mathbb{C}$. Define $\tau = \omega_1/\omega_2$ and the lattice

$$\Lambda_\tau := \omega_2^{-1}\Lambda = \tau\mathbb{Z} \oplus \mathbb{Z}.$$

Then by Corollary 4.2 we have that $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda_\tau$ so it suffices to prove the theorem for $\text{End}(\mathbb{C}/\Lambda_\tau)$. This will be done by constructing an order $R \subset \mathbb{Q}(\tau)$ such that

$\text{End}(\mathbb{C}/\Lambda_\tau) \cong R$ and showing that if $R \neq \mathbb{Z}$ the field $\mathbb{Q}(\tau)$ is quadratic imaginary. Let

$$R = \{\alpha \in \mathbb{C} \mid \alpha\Lambda_\tau \subseteq \Lambda_\tau\}.$$

It is clear that $R$ is closed under addition and multiplication, so it is a ring. Once again from Corollary 4.2 it follows that $R \cong \text{End}(\mathbb{C}/\Lambda_\tau)$.

Let $r$ be a nonzero element of $r$. Since $\tau, 1 \in \Lambda_\tau$ the definition of $R$ ensures that there exist $a, b, c, d \in \mathbb{Z}$ such that $r\tau = a\tau + b$ and $r \cdot 1 = c\tau + d$. Since $r \neq 0$ we may divide the first equation by the second to eliminate $r$ from the system, producing

$$c\tau^2 - (a - d)\tau - b = 0.$$

Now assume $R \neq \mathbb{Z}$. Then $c \neq 0$ and since $\tau \notin \mathbb{R}$, this implies that $\mathbb{Q}(\tau)$ is quadratic imaginary.

We now show that $R$ is in fact an order in $\mathbb{Q}(\tau)$. In matrix form, the system of equations provided above is

$$\begin{pmatrix} r - a & -b \\ -c & r - d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0.$$

This in turn means

$$\det \begin{pmatrix} r - a & -b \\ -c & r - d \end{pmatrix} = 0,$$

which is a monic polynomial with integer coefficients for which $r$ is a zero. Thus, $r$ is integral. In addition, since for every $r \in R$ there exist $c, d \in \mathbb{Z}$ such that $r = c\tau + d$ we see that $R \subseteq \mathbb{Q}(\tau)$ and hence $R$ is a subring of $\mathcal{O}_{\mathbb{Q}(\tau)}$. Since $R \neq \mathbb{Z}$ we may let $r \in R \setminus \mathbb{Z}$. This means that also $r \in \mathbb{Q}(\tau) \setminus \mathbb{Z}$ and so $r\mathbb{Z} \oplus \mathbb{Z}$ is a free abelian group of rank 2. Then, the inclusions $r\mathbb{Z} \oplus \mathbb{Z} \subseteq R \subseteq \mathbb{Q}(\tau)$ imply that $R$ is of rank 2. Thus, it is an order in $\mathbb{Q}(\tau)$. □

If a complex torus $\mathbb{C}/\Lambda$ has an endomorphism ring isomorphic to $R \supset \mathbb{Z}$ we say that $\mathbb{C}/\Lambda$ has complex multiplication by $R$.

## 4.2 The Functions $\sigma, \zeta, \wp$, and $\eta$

We begin by studying a few functions whose properties will form the basis of understanding the Siegel functions. The exposition here mainly follows [10] and [16]. Let $\Lambda$ be a lattice and define the Weierstrass $\sigma$ function as

$$\sigma(z, \Lambda) = z \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}. \tag{4.1}$$

This function is an example of a Weierstrass product. In particular, the product converges uniformly on compact subsets of $\mathbb{C}$, $\sigma$ is holomorphic in $z$ on the entire complex plane, and $\sigma$ has simple zeros precisely at the elements of $\Lambda$. Moreover, uniform convergence of the product ensures that we are able to calculate the logarithmic derivative of $\sigma$ term by term, and the resulting series converges uniformly on compact subsets not containing elements of $\Lambda$. Proofs of these facts can be found in a standard introductory complex analysis book such as [20, Ch 5]. Moreover, note that $\sigma$ is homogeneous of degree 1, meaning

$$\sigma(mz, m\Lambda) = m\sigma(z, \Lambda), \tag{4.2}$$

for all $m \in \mathbb{C}^{\times}$. This follows immediately from the definition and noting that the product converges absolutely.

Now, due to the remarks above, we may take the logarithmic derivative of $\sigma$ as follows, which produces the Weierstrass zeta function

$$\zeta(z, \Lambda) := \frac{\sigma'(z, \Lambda)}{\sigma(z, \Lambda)} = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right].$$

From this definition it is immediate that $\zeta$ is homogeneous of degree $-1$, satisfying the relation

$$\zeta(mz, m\Lambda) = m^{-1}\zeta(z, \Lambda) \tag{4.3}$$

for all $m \in \mathbb{C}^{\times}$. Moreover, as mentioned in the comments above, the series defining $\zeta$ converges uniformly on all compact subsets not containing an element of $\Lambda$. Thus, we may differentiate $\zeta$ term by term (see [20, Ch 2, Thm 5.3]) and define the Weierstrass

$\wp$ function as

$$\wp(z,\Lambda) = -\zeta'(z,\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]. \tag{4.4}$$

We also may obtain its derivative

$$\wp'(z,\Lambda) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}.$$

Each of these functions does depend on the lattice chosen. For ease of notation, however, they will often be denoted simply as functions of $z$ when the lattice being used is clear. We have the following important property of the $\wp$ function.

**Lemma 4.4.** *The $\wp$ function is invariant under translation by elements of $\Lambda$. That is, $\wp(z+\omega) = \wp(z)$ for all $\omega \in \Lambda$.*

*Proof.* Assume $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. First notice that $\wp'(z + w_i) = \wp'(z)$, since the summation defining $\wp'(z)$ converges absolutely, and $\wp'(z+w_i)$ is just a rearrangement of $\wp'(z)$. This means there exists a constant $c_i$ such that $\wp(z+w_i) = \wp(z)+c_i$. Letting $z = -\frac{w_i}{2}$ in this expression and using the fact that $\wp(z)$ is even implies that $c_i = 0$.

Similarly, there exists a constant $d_i$ such that $\wp(z - w_i) = \wp(z) + d_i$, and one concludes that $d_i = 0$ by letting $z = \frac{w_i}{2}$. Together, these observations show that $\wp(z)$ is translation invariant for all $\omega \in \Lambda$. □

This lemma along with (4.4) imply that

$$\zeta'(z+\omega) - \zeta'(z) = \wp(z+\omega) - \wp(z) = 0$$

for all $\omega \in \Lambda$. Taking the antiderivative shows that there is a map $\eta : \Lambda \to \mathbb{C}$ which is independent of $z$ such that

$$\zeta(z+\omega) = \zeta(z) + \eta(\omega).$$

This map is $\mathbb{Z}$-linear, which can be seen as follows. Let $\lambda_1, \lambda_2 \in \Lambda$. Then on one hand

$$\zeta(z+\lambda_1+\lambda_2) = \zeta(z) + \eta(\lambda_1+\lambda_2),$$

and on the other hand

$$\zeta(z+\lambda_1+\lambda_2)=\zeta(z+\lambda_1)+\eta(\lambda_2)=\zeta(z)+\eta(\lambda_1)+\eta(\lambda_2).$$

This shows that $\eta$ is $\mathbb{Z}$-linear. For notational simplicity, given a lattice $\Lambda=\omega_1\mathbb{Z}\oplus\omega_2\mathbb{Z}$ with $\frac{\omega_1}{\omega_2}\in\mathbb{H}$ define

$$\eta_1:=\eta(\omega_1)\quad\text{and}\quad\eta_2:=\eta(\omega_2).$$

The map $\eta$ can be extended to an $\mathbb{R}$-linear map on all of $\mathbb{C}$, which will also be referred to as $\eta$. To do this, write $z\in\mathbb{C}$ as $z=t_1\omega_1+t_2\omega_2$. Then define

$$\eta(z):=t_1\eta_1+t_2\eta_2.$$

This map $\eta:\mathbb{C}\to\mathbb{C}$ is called the Weierstrass $\eta$ function. By using (4.3) it can be seen that the Weierstrass $\eta$ function is homogeneous of degree $-1$. To see this, first observe that the function $\eta:\Lambda\to\mathbb{C}$ is homogeneous of degree $-1$. Indeed, given $m\in\mathbb{C}^\times$ and $\omega\in\Lambda$ it follows that

$$\begin{aligned}\zeta(mz+m\omega,m\Lambda)&=\zeta(mz,m\Lambda)+\eta(m\omega,m\Lambda)\\&=m^{-1}\zeta(z,\Lambda)+\eta(m\omega,m\Lambda).\end{aligned}$$

However, also

$$\begin{aligned}\zeta(mz+m\omega,m\Lambda)&=m^{-1}\zeta(z+\omega,\Lambda)\\&=m^{-1}\zeta(z,\Lambda)+m^{-1}\eta(\omega,\Lambda).\end{aligned}$$

These two results imply that if $\omega\in\Lambda$ then

$$\eta(m\omega,m\Lambda)=m^{-1}\eta(\omega,\Lambda)$$

for all $m\in\mathbb{C}^\times$. Now, assume $\Lambda=\omega_1\mathbb{Z}\oplus\omega_2\mathbb{Z}$ and let $z=t_1\omega_1+t_2\omega_2$ be an arbitrary complex number. Then, using the definition of $\eta:\mathbb{C}\to\mathbb{C}$ along with the fact that

$\eta : \Lambda \to \mathbb{C}$ is homogeneous of degree $-1$ it follows that

$$\begin{aligned}\eta(mz, m\Lambda) &= t_1\eta(m\omega_1, m\Lambda) + t_2\eta(m\omega_2, m\Lambda) \\ &= m^{-1}t_1\eta(\omega_1, \Lambda) + m^{-1}t_2\eta(\omega_2, \Lambda) \\ &= m^{-1}\eta(z, \Lambda),\end{aligned} \tag{4.5}$$

for all $m \in \mathbb{C}^\times$.

**Theorem 4.5** (Legendre Relation)**.** *If* $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ *is a lattice with* $\omega_1/\omega_2 \in \mathbb{H}$*, then* $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$*.*

*Proof.* This follows immediately from integrating $\zeta$ around appropriate contour in the complex plane. Given $z \in \mathbb{C}$, consider the parallelogram

$$P_z = \{z + a_1\omega_1 + a_2\omega_2 \mid a_1, a_2 \in [0, 1]\}.$$

Choose $\gamma \in \mathbb{C}$ such that 0 is in the interior of $P_\gamma$. Then, because $\zeta$ is holomorphic everywhere apart from a simple pole at $z = 0$ with residue 1, it follows that

$$\oint_{\partial P_\gamma} \zeta(z)dz = 2\pi i,$$

so long as the path along $\partial P_\gamma$ is taken to be counterclockwise. That is, we integrate along straight lines first from $\gamma$ to $\gamma + \omega_2$, next from $\gamma + \omega_2$ to $\gamma + \omega_1 + \omega_2$, next from $\gamma + \omega_1 + \omega_2$ to $\gamma + \omega_1$, and finally from $\gamma + \omega_1$ to $\gamma$. Breaking the integral up by the sides of the parallelogram in this way produces

$$2\pi i = \oint_{\partial P_\gamma} \zeta(z)dz = I_1 + I_2 + I_3 + I_4, \tag{4.6}$$

where

$$\begin{aligned}
I_1 &= \int_{\gamma}^{\gamma+\omega_2} \zeta(z)dz = \int_0^1 \zeta(\gamma + t\omega_2)\omega_2 dt \\
I_2 &= \int_{\gamma+\omega_2}^{\gamma+\omega_1+\omega_2} \zeta(z)dz = \int_0^1 \zeta(\gamma + \omega_2 + t\omega_1)\omega_1 dt \\
-I_3 &= \int_{\gamma+\omega_1}^{\gamma+\omega_1+\omega_2} \zeta(z)dz = \int_0^1 \zeta(\gamma + \omega_1 + t\omega_2)\omega_2 dt \\
-I_4 &= \int_{\gamma}^{\gamma+\omega_1} \zeta(z)dz = \int_0^1 \zeta(\gamma + t\omega_1)\omega_1 dt.
\end{aligned}$$

Using the additivity of $\eta$ we see that

$$I_1 + I_3 = -\int_0^1 \eta_1\omega_2 dt = -\omega_2\eta_1$$

and

$$I_2 + I_4 = \int_0^1 \eta_2\omega_1 dt = \omega_1\eta_2.$$

These expressions along with (4.6) immediately give the desired result. □

**Theorem 4.6.** *Let $\Lambda$ be a lattice and $\omega \in \Lambda$. The Weierstrass $\sigma$ function satisfies the following functional equation:*

$$\sigma(z + \omega) = \psi(\omega)e^{\eta(\omega)(z+\omega/2)}\sigma(z), \tag{4.7}$$

*where*

$$\psi(\omega) = \begin{cases} 1, & \frac{\omega}{2} \in \Lambda; \\ -1 & \frac{\omega}{2} \notin \Lambda. \end{cases}$$

*Proof.* Define the function $f : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ via

$$f(z) = \frac{\sigma(z + \omega)}{\sigma(z)}.$$

Note that since the zeros of $\sigma$ are all simple zeros and occur precisely at the lattice points, $f$ can be extended to a holomorphic and non-vanishing function on all of $\mathbb{C}$. Therefore, there exists a holomorphic function $h : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = e^{h(z)}. \tag{4.8}$$

Moreover, let $D$ denote the logarithmic differentiation operator. That is, for a complex valued function $g$, we have $D(g) = g'/g$. Applying this operator to both sides of (4.8) produces

$$D(f(z)) = D(e^{h(z)}) = h'(z).$$

On the other hand, explicitly calculating $D(f)$ produces

$$\begin{aligned} D\left(\frac{\sigma(z+\omega)}{\sigma(z)}\right) &= \frac{\sigma(z)}{\sigma(z+\omega)}\left[\frac{\sigma'(z+\omega)\sigma(z) - \sigma(z+\omega)\sigma'(z)}{[\sigma(z)]^2}\right] \\ &= \frac{\sigma'(z+\omega)}{\sigma(z+\omega)} - \frac{\sigma'(z)}{\sigma(z)} \\ &= \zeta(z+\omega) - \zeta(z) \\ &= \eta(\omega). \end{aligned}$$

Thus, it follows that $h'(z) = \eta(\omega)$ and taking the antiderivative shows that

$$h(z) = \eta(\omega)z + C(\omega)$$

where $C(\omega)$ is independent of $z$. Substituting this expression into (4.8) implies that

$$\sigma(z+\omega) = e^{\eta(\omega)z + C(\omega)}\sigma(z).$$

Thus, by choosing the appropriate constant $\psi(\omega)$, the expression for $\sigma(z+\omega)$ may be rewritten in the form given by (4.7). It remains to determine that $\psi(\omega)$ has the form specified in the statement of the theorem.

To do this, first assume $\frac{\omega}{2} \notin \Lambda$. This means that $\sigma\left(-\frac{\omega}{2}\right) \neq 0$, so substituting $z = -\frac{\omega}{2}$ into (4.7) implies

$$\psi(\omega) = \frac{\sigma\left(\frac{\omega}{2}\right)}{\sigma\left(-\frac{\omega}{2}\right)} = \frac{\sigma\left(\frac{\omega}{2}, \Lambda\right)}{\sigma\left(-\frac{\omega}{2}, \Lambda\right)} = -1, \qquad (4.9)$$

where the final equality follows because $\sigma$ is homogeneous of degree 1 and $\Lambda = -\Lambda$.

The situation when $\frac{\omega}{2} \in \Lambda$ is slightly more complicated. First notice that for any $\omega \in \Lambda$ we have

$$\frac{\sigma(z+2\omega)}{\sigma(z)} = \frac{\sigma(z+2\omega)}{\sigma(z+\omega)} \cdot \frac{\sigma(z+\omega)}{\sigma(z)}.$$

Applying (4.7) to rewrite each of the fractions gives

$$\psi(2\omega)e^{\eta(2\omega)(z+\omega)} = \psi(\omega)e^{\eta(\omega)(z+\omega+\omega/2)} \cdot \psi(\omega)e^{\eta(\omega)(z+\omega/2)}.$$

Linearity of $\eta$ implies that $\eta(2\omega) = 2\eta(\omega)$, so the exponentials cancel and

$$\psi(2\omega) = [\psi(\omega)]^2 \tag{4.10}$$

for all $\omega \in \Lambda$. Finally, consider $k$ the smallest natural number such that $\frac{\omega}{2^{k+1}} \notin \Lambda$. Then (4.9) implies

$$\psi\left(\frac{\omega}{2^k}\right) = -1.$$

Moreover, by (4.10) it follows that

$$\psi\left(\frac{\omega}{2^{n-1}}\right) = \left[\psi\left(\frac{\omega}{2^n}\right)\right]^2$$

for all natural $n \leq k$. Applying this formula recursively gives

$$\psi(\omega) = \left[\psi\left(\frac{\omega}{2^k}\right)\right]^{2^k}.$$

Since $\frac{\omega}{2} \in \Lambda$ this means $k \geq 1$ and we have that

$$\psi(\omega) = (-1)^{2^k} = 1,$$

as desired. □

**Corollary 4.7.** *If* $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ *and* $\omega \in \Lambda$ *is given by* $\omega = b_1\omega_1 + b_2\omega_2$ *then*

$$\psi(\omega) = -(-1)^{(b_1+1)(b_2+1)}.$$

*Proof.* If $\omega/2 \in \Lambda$ then both $b_1$ and $b_2$ are even, so clearly $(b_1+1)(b_2+1)$ is odd and $-(-1)^{(b_1+1)(b_2+1)} = 1$. On the other hand, if $\omega/2 \notin \Lambda$ then either $b_1$ or $b_2$ is odd. Thus, $(b_1+1)(b_2+1)$ is even and $-(-1)^{(b_1+1)(b_2+1)} = -1$. □

To close the section a useful product expansion for $\sigma(z, \Lambda_\tau)$ is presented. First, we need a lemma.

**Lemma 4.8.** *Let $\tau \in \mathbb{H}$. Define the function*

$$g(z, \Lambda_\tau) = e^{z^2\eta_2/2}(q_z^{1/2} - q_z^{-1/2})P_\tau(z),$$

*where $P_\tau(z) = \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1})$. Then $g(z)$ satisfies the following relationships:*

$$g(z+1) = -e^{\eta_2(z+1/2)}g(z)$$
$$g(z+\tau) = -e^{\eta_1(z+\tau/2)}g(z).$$

*Proof.* The identity for $g(z+1)$ follows immediately by noting that $q_{z+1}^{\pm 1/2} = -q_z^{\pm 1/2}$ and $q_{z+1} = q_z$. For the other identity, begin by noticing that $q_{z+\tau}^{\pm 1/2} = q_\tau^{\pm 1/2} q_z^{\pm 1/2}$. Thus,

$$\begin{aligned} g(z+\tau) &= e^{(z+\tau)^2\eta_2/2}((q_\tau q_z)^{1/2} - (q_\tau q_z)^{-1/2}) \prod_{n=1}^{\infty}(1 - q_\tau^{n+1}q_z)(1 - q_\tau^{n-1}q_z^{-1}) \\ &= -e^{(z+\tau)^2\eta_2/2}(q_\tau q_z)^{-1/2}(1 - q_\tau q_z) \prod_{n=1}^{\infty}(1 - q_\tau^{n+1}q_z)(1 - q_\tau^{n-1}q_z^{-1}). \end{aligned}$$

Comparing $P_\tau(z)$ and $P_\tau(z+\tau)$, one can see that

$$P_\tau(z+\tau) = \frac{1 - q_z^{-1}}{1 - q_\tau q_z} \cdot P_\tau(z).$$

This implies

$$\begin{aligned} g(z+\tau) &= -e^{(z+\tau)^2\eta_2/2}(q_\tau q_z)^{-1/2}(1 - q_z^{-1})P_\tau(z) \\ &= -e^{(z+\tau)^2\eta_2/2}q_\tau^{-1/2}q_z^{-1}(q_z^{1/2} - q_z^{-1/2})P_\tau(z) \\ &= -e^{\eta_2(2\tau z+\tau^2)/2}q_\tau^{-1/2}q_z^{-1}g(z). \end{aligned}$$

By combining the factors preceding $g(z)$ in this expression, one sees that the exponent is given by

$$\frac{\eta_2}{2}(2\tau z + \tau^2) - \pi i\tau - 2\pi iz = \left(\frac{\tau}{2} + z\right)(\eta_2\tau - 2\pi i).$$

Now, since $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$ recall that the Legendre relation from Theorem 4.5 in this case states $\tau\eta_2 - \eta_1 = 2\pi i$. Applying this to the expression above immediately implies the desired identity for $g(z+\tau)$. □

**Theorem 4.9.** *Given $\tau \in \mathbb{H}$ and $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$ the Weierstrass $\sigma$ function has the following product expansion*

$$\sigma(z, \Lambda_\tau) = \frac{1}{2\pi i} e^{z^2\eta_2/2}(q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty} \frac{(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1})}{(1 - q_\tau^n)^2}. \tag{4.11}$$

*Proof.* The function $g(z)$ defined in the previous lemma is a holomorphic function in $z$ with the same zeros as $\sigma(z)$. Moreover, all zeros of both $\sigma(z)$ and $g(z)$ are of the same order (in fact, they are all simple). Therefore, the function $f(z) = \frac{\sigma(z)}{g(z)}$ is holomorphic. It will now be shown that, in addition, $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda_\tau$.

It suffices to show that $f(z \pm \tau) = f(z)$ and $f(z \pm 1) = f(z)$. However, since both $\sigma(z)$ and $g(z)$ are odd functions, $f(z)$ is even, so in fact it suffices to prove that $f(z+\tau) = f(z)$ and $f(z+1) = f(z)$. Using Lemma 4.8 as well as (4.7) with $\omega = \tau$ and $\Lambda = \Lambda_\tau$ we have

$$f(z+\tau) = \frac{-e^{\eta_1(z+\tau/2)}\sigma(z)}{-e^{\eta_1(z+\tau/2)}g(z)} = f(z).$$

Following the same strategy, but letting $\omega = 1$ in (4.7) instead produces

$$f(z+1) = \frac{-e^{\eta_2(z+1/2)}\sigma(z)}{-e^{\eta_2(z+1/2)}g(z)} = f(z).$$

This shows that $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda_\tau$. Thus $f(z)$ is in fact bounded, and so it is constant as a function of $z$ by Liouville's theorem. This constant is determined by examining the limit as $z \to 0$. Using the definition of $\sigma(z)$ from (4.1) one can see that

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{z\to 0} \left( \frac{z}{q_z^{1/2} - q_z^{-1/2}} \right) \cdot \frac{1}{\prod_{n=1}^{\infty}(1 - q_\tau^n)^2} \\ &= \frac{1}{2\pi i \prod_{n=1}^{\infty}(1 - q_\tau^n)^2}. \end{aligned}$$

Thus, it immediately follows that

$$\sigma(z) = \frac{g(z)}{2\pi i \prod_{n=1}^{\infty}(1-q_\tau^n)^2},$$

which completes the proof.

□

## 4.3 Automorphic Forms Defined

We begin by recalling some basic definitions and facts from the theory of automorphic forms. Firstly, define the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}.$$

This group acts on the upper half plane via the linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}.$$

This is easily verified from the standard formula

$$\mathrm{Im}(\alpha\tau) = \frac{\mathrm{Im}(\tau)\det(\alpha)}{|c\tau+d|^2}, \tag{4.12}$$

which holds for all $2\times 2$ matrices $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with real entries. Two subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which will play an important role in what follows are $\Gamma(N)$ and $\Gamma_1(N)$. They are defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N \right\}.$$

In general, a subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup of level $N$ if $\Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{N}$. In particular, both $\Gamma(N)$ and $\Gamma_1(N)$ are congruence subgroups of level $N$.

To define automorphic forms on a congruence subgroup $\Gamma$, one must first discuss what it means for a meromorphic function on the upper half plane to be weakly modular, as well as what it means for such a function to be meromorphic at infinity. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, let $f : \mathbb{H} \to \mathbb{C}$, and let $k \in \mathbb{Z}$. Define the weight $k$ operator $[\alpha]_k$ by the rule

$$f|_{[\alpha]_k}(\tau) := (c\tau + d)^{-k} f(\alpha\tau).$$

If $\Gamma$ is a congruence subgroup, a function $f : \mathbb{H} \to \mathbb{C}$ is called weight $k$ invariant under $\Gamma$ if

$$f|_{[\gamma]_k} = f$$

for all $\gamma \in \Gamma$. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ that is weight $k$ invariant for $\Gamma$ is called weakly modular of weight $k$ for $\Gamma$.

Let $f : \mathbb{H} \to \mathbb{C}$ be weakly modular of weight $k$ with respect to $\Gamma$. Let $h$ be the smallest positive integer such that

$$\gamma_h := \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Since $\Gamma(N) \subseteq \Gamma$ such an $h$ is guaranteed to exist. Moreover $h \leq N$. Note that for all $n \in \mathbb{Z}$

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix}.$$

Applying the condition $f|_{[\gamma_h^n]} = f$ implies that $f(\tau + nh) = f(\tau)$ for all $n \in \mathbb{Z}$. Because of this property, $f$ can be used to define a new function $g$ on a punctured disk about 0. This works as follows.

Assume there exists a $c_0 > 0$ such that $f$ is holomorphic on the domain

$$\mathbb{H}_{c_0} = \{z \in \mathbb{H} \mid \mathrm{Im}(z) > c_0\}.$$

Given $\tau \in \mathbb{H}_{c_0}$ the map $\tau \mapsto e^{2\pi i\tau/h}$ sends $\mathbb{H}_{c_0}$ to a punctured disk $D$ centered at the origin. Define the function $g : D \to \mathbb{C}$ via the rule

$$g(q) = f\left(\frac{h\log(q)}{2\pi i}\right).$$

Though the logarithm is multivalued, $g$ is well defined due to the $h$-periodicity of $f$, and in fact, it can be shown that $g$ is holomorphic on $D$. Moreover,

$$g(e^{2\pi i\tau/h}) = f(\tau + nh) = f(\tau).$$

It follows that $f$ has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q_\tau^{n/h}, \qquad q_\tau = e^{2\pi i\tau}.$$

As $q_\tau \to 0$ we have that $\mathrm{Im}(\tau) \to \infty$. This notion leads to the following definition.

**Definition 4.10.** A weakly modular function $f$ is said to be meromorphic at infinity if

(1) There exists a $c_0 > 0$ such that $f$ is holomorphic on the domain

$$\{z \in \mathbb{C} \mid \mathrm{Im}(z) > c_0\}.$$

(2) The Fourier series expansion for $f(\tau)$ has the form

$$f(\tau) = \sum_{n=n_0}^{\infty} a_n q_\tau^{n/h}, \qquad \mathrm{Im}(\tau) > c_0$$

for some $n_0 \in \mathbb{Z}$.

If in fact $n_0 \geq 0$ then $f$ is said to be holomorphic at infinity, and if $n_0 > 0$ then $f$ is said to vanish at infinity. Three related sets of forms are now defined.

**Definition 4.11.** Let $\Gamma$ be a congruence subgroup.

(1) The set of automorphic forms of weight $k$ for $\Gamma$ is denoted $\mathcal{A}_k(\Gamma)$ and consists of all the weakly modular functions $f$ for which $f|_{[\alpha]_k}$ is meromorphic at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

(2) The set of modular forms of weight $k$ for $\Gamma$ is denoted $\mathcal{M}_k(\Gamma)$ and consists of all the automorphic forms $f$ which are holomorphic and for which $f|_{[\alpha]_k}$ is holomorphic at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

(3) The set of cusp forms of weight $k$ for $\Gamma$ is denoted $\mathcal{S}_k(\Gamma)$ and consists of all the modular forms $f$ for which $f|_{[\alpha]_k}$ vanishes at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

It is immediate from these definitions that

$$\mathcal{S}_k(\Gamma) \subseteq \mathcal{M}_k(\Gamma) \subseteq \mathcal{A}_k(\Gamma).$$

Though automorphic forms are defined on the upper half plane, there is another perspective that can be taken, which will be quite useful in what follows. Let $W \subseteq \mathbb{C}^2$ consist of all column vectors $(\omega_1, \omega_2)^t$ of complex numbers such that $\frac{\omega_1}{\omega_2} \in \mathbb{H}$. We begin by establishing a bijection between functions $f : \mathbb{H} \to \mathbb{C}$ and homogeneous functions $F : W \to \mathbb{C}$ of degree $k$.

Given $f : \mathbb{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$ define a homogeneous function $F$ of degree $-k$ on $W$ via the rule

$$F\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} := \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right).$$

Conversely, if $F : W \to \mathbb{C}$ is homogeneous of degree $-k$ then define a function $f : \mathbb{H} \to \mathbb{C}$ via the rule

$$f(\tau) := F\begin{pmatrix}\tau\\ 1\end{pmatrix}.$$

This establishes the desired bijection. The function $F : W \to \mathbb{C}$ is referred to as the homogeneous function corresponding to $f$. If $f \in \mathcal{A}_k(\Gamma)$ for some congruence subgroup $\Gamma$, then the function $F : W \to \mathbb{C}$ is referred to as the homogeneous automorphic form corresponding to $f$. We have the following useful theorem.

**Theorem 4.12.** *A function $f : \mathbb{H} \to \mathbb{C}$ satisfies $f|_{[\gamma]_k} = f$ for all $\gamma \in \Gamma$ if and only if the corresponding homogeneous function $F$ of degree $-k$ satisfies*

$$F\left(\gamma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right) = F\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \tag{4.13}$$

*for all $\gamma \in \Gamma$.*

*Proof.* Let $f : \mathbb{H} \to \mathbb{C}$ satisfy $f|_{[\gamma]_k} = f$ for all $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$. Defining $\tau = \frac{\omega_1}{\omega_2}$, we have

$$F\left(\gamma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right) = F\begin{pmatrix}a\omega_1 + b\omega_2\\ c\omega_1 + d\omega_2\end{pmatrix}$$
$$= \omega_2^{-k} F\begin{pmatrix}a\tau + b\\ c\tau + d\end{pmatrix}$$
$$= \omega_2^{-k}(c\tau + d)^{-k} f(\gamma\tau)$$

But, $f|_{[\gamma]_k}(\tau) = (c\tau + d)^{-k} f(\gamma\tau) = f(\tau)$. Thus

$$F\left(\gamma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right) = \omega_2^{-k} f(\tau)$$
$$= F\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Conversely, let $F : W \to \mathbb{C}$ be homogeneous of degree $-k$, satisfying (4.13). Then

$$f|_{[\gamma]_k}(\tau) = (c\tau + d)^{-k} f(\gamma\tau)$$
$$= (c\tau + d)^{-k} F\begin{pmatrix}\gamma\tau\\ 1\end{pmatrix}$$
$$= (c\tau + d)^{-k} F\left((c\tau + d)^{-1}\begin{pmatrix}a\tau + b\\ c\tau + d\end{pmatrix}\right)$$
$$= F\begin{pmatrix}a\tau + b\\ c\tau + d\end{pmatrix}.$$

Since $\left(\begin{smallmatrix}a\tau+b\\ c\tau+d\end{smallmatrix}\right) = \gamma\left(\begin{smallmatrix}\tau\\ 1\end{smallmatrix}\right)$, equation (4.13) immediately implies that $f|_{[\gamma]_k}(\tau) = f(\tau)$, completing the proof. □

In light of this theorem, a homogeneous function $F : W \to \mathbb{C}$ of degree $-k$ is said to be an automorphic form of weight $k$ for $\Gamma$ if it satisfies (4.13), and if the corresponding function $f : \mathbb{H} \to \mathbb{C}$ satisfies the proper conditions on meromorphicity outlined in Definition 4.11. Moreover, if a function $F : W \to \mathbb{C}$ is said to be meromorphic or holomorphic, this should be assumed to mean that the corresponding function $f : \mathbb{H} \to \mathbb{C}$ is meromorphic or holomorphic, respectively.

Note that each element of $W$ defines a basis of a lattice, but in general such a lattice has many different bases. Thus, we recall a useful theorem which describes the relationship between any two bases of a lattice.

**Theorem 4.13.** *Let* $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ *and* $\Lambda' = \omega_1'\mathbb{Z} \oplus \omega_2'\mathbb{Z}$ *be lattices in* $\mathbb{C}$ *with* $\omega_1/\omega_2 \in \mathbb{H}$ *and* $\omega_1'/\omega_2' \in \mathbb{H}$*. Then* $\Lambda = \Lambda'$ *if and only if*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_2 \end{pmatrix} \tag{4.14}$$

*for some* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*.*

*Proof.* Assume there exists a matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ satisfying (4.14). Then if $\lambda' \in \Lambda'$ there exist $m, n \in \mathbb{Z}$ such that $\lambda' = m\omega_1' + n\omega_2'$. But (4.14) shows that both $\omega_1'$ and $\omega_2'$ can be written as linear combinations of $\omega_1$ and $\omega_2$ using integer coefficients. Thus, $\lambda'$ can be written in this way as well, so $\lambda' \in \Lambda$. Similarly, if $\lambda \in \Lambda$ there exist $m, n \in \mathbb{Z}$ (possibly different than before) such that $\lambda = m\omega_1 + n\omega_2$. Because $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ it follows that $\alpha^{-1} \in \mathrm{SL}_2(\mathbb{Z})$ and thus has integer entries. This along with (4.14) implies that both $\omega_1$ and $\omega_2$ can be written as linear combinations of $\omega_1'$ and $\omega_2'$ with integer coefficients. Thus, $\lambda$ can also be written in this way, so $\lambda \in \Lambda'$. This establishes that $\Lambda = \Lambda'$.

Conversely, assume that $\Lambda = \Lambda'$. Then, there exists a matrix $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with integer entries such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Now let $\tau = \omega_1/\omega_2$ and note that $\frac{\omega_1'}{\omega_2'} = \alpha\tau$. Equation (4.12) implies that $\det(\alpha) > 0$ since $\mathrm{Im}(\omega_1'/\omega_2') > 0$ and $\mathrm{Im}(\omega_1/\omega_2) > 0$. Thus, $\alpha$ is invertible. Using this, and letting $D = \det(\alpha)$ we find

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

Because $\Lambda = \Lambda'$ then $\{\omega_1', \omega_2'\}$ is a basis for $\Lambda$ and each element of $\Lambda$ is represented uniquely as a linear combination of $\omega_1'$ and $\omega_2'$ with integer coefficients. Thus, each entry of the matrix $\frac{1}{D}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ is an integer. This means that $D$ divides each of the entries of $\alpha$. Thus $D^2 \mid ad - bc$. This can only occur if $D = \pm 1$ but it was already noted that $D > 0$. Thus $D = 1$ so $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. This establishes that (4.14) is satisfied if $\Lambda = \Lambda'$. $\square$

### 4.3.1 The Modular Forms $G_k$ and $\Delta$

In what follows, if $\Omega = (\omega_1, \omega_2)^t \in \mathbb{C}^2$ with $\frac{\omega_1}{\omega_2} \in \mathbb{H}$ then the lattice with $\{\omega_1, \omega_2\}$ as a basis will be denoted $\Lambda_\Omega$. Note by Theorem 4.13 that $\Lambda_{\alpha\Omega} = \Lambda_\Omega$ if and only if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. If $\tau \in \mathbb{H}$ then $\Lambda_\tau$ will be used to denote the lattice with $\{\tau, 1\}$ as a basis.

Let $\Omega = (\omega_1, \omega_2)^t$ with $\frac{\omega_1}{\omega_2} \in \mathbb{H}$. Let $k > 2$ be even and define the Eisenstein series of weight $k$ by

$$G_k(\Omega) = \sum_{\omega \in \Lambda_\Omega}{}' \frac{1}{\omega^k},$$

where the primed summation indicates that this is a sum over only nonzero $\omega$. This series defines a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $k$. Indeed, inspecting the series immediately shows that $G_k(\Omega)$ is homogeneous of degree $-k$. Moreover, let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and recall that by Theorem 4.13 we have $\Lambda_\Omega = \Lambda_{\alpha\Omega}$. Thus, it follows that $G_k(\alpha\Omega) = G_k(\Omega)$.

It remains to show that $G_k(\tau)$ is holomorphic and that $G_k|_{[\alpha]_k}(\tau)$ is holomorphic at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Since $k > 2$ the sum defining $G_k(\tau)$ converges absolutely, and converges uniformly on compact subsets of $\mathbb{C}$. Thus, $G_k(\tau)$ is holomorphic. Moreover, note that by using the Fourier expansion of the cotangent function, it can be shown that $G_k(\tau)$ also has a Fourier expansion for all $\tau \in \mathbb{H}$, and moreover that $G_k(\tau)$ is holomorphic at infinity. See any introductory text on modular forms, such as [5] for this derivation. Finally, because $G_k(\Omega)$ is homogeneous of degree $-k$ and invariant under $\mathrm{SL}_2(\mathbb{Z})$ this means $G_k(\tau)$ satisfies $G_k|_{[\alpha]_k} = G_k$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. So since $G_k(\tau)$ is holomorphic at infinity, so is $G_k|_{[\alpha]_k}$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

We now define the discriminant function $\Delta$, an important cusp form that will be used in the sequel. First, define

$$g_4(\tau) = 60G_4(\tau) \quad g_6(\tau) = 140G_6(\tau).$$

Then the discriminant function is defined by

$$\Delta(\tau) = (g_4(\tau))^3 - 27(g_6(\tau))^2. \tag{4.15}$$

The coefficients 60 and 140 from the definitions of $g_4$ and $g_6$ ensure that the Fourier expansion of $\Delta(\tau)$ has leading coefficient 0. This is easily verified from the form of the Fourier series for $G_k(\tau)$, which is well known. This fact along with the previous discussion on the Eisenstein series immediately produces the following theorem.

**Theorem 4.14.** *The discriminant function $\Delta$ is a cusp form of weight 12 on $SL_2(\mathbb{Z})$.*

We also remark before moving on that $\Delta$ can be written in the form of an infinite product. First, define the Dedekind $\eta$ function via the infinite product

$$\eta(\tau) = q_\tau^{1/24} \prod_{n=1}^{\infty} (1 - q_\tau^n), \tag{4.16}$$

where $q_\tau = e^{2\pi i \tau}$. Note that the $\eta$ used here is distinct from the Weierstrass $\eta$ function introduced previously in Section 4.2. We have the following identity:

$$\Delta = (2\pi)^{12} \eta^{24}. \tag{4.17}$$

This can be deduced by proving that $\mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is a one dimensional vector space over $\mathbb{C}$ containing both $\eta^{24}$ and $\Delta$, and then comparing the leading coefficients of the Fourier series for $\Delta$ and $\eta^{24}$. The details can be found in an introductory text to modular forms such as [5].

## 4.4 Automorphic Forms for $\Gamma_1(N)$

As previously discussed, an automorphic form for a congruence subgroup $\Gamma$ can be thought of either as a function on the upper half plane, or as a function on $W$, the set of column vectors $(\omega_1, \omega_2) \in \mathbb{C}^2$ for which $\frac{\omega_1}{\omega_2} \in \mathbb{H}$. In the case where $\Gamma = \Gamma_1(N)$ two more useful perspectives with which to view automorphic forms arise. The first involves viewing an automorphic form as a function on enhanced elliptic curves, which will now be described.

**Definition 4.15.** Let $\Lambda$ be a lattice in $\mathbb{C}$ and $p+\Lambda$ be a point of order $N$ on the elliptic curve $\mathbb{C}/\Lambda$. The pair $(p+\Lambda, \mathbb{C}/\Lambda)$ is called an enhanced elliptic curve.

**Definition 4.16.** A complex valued function $\widetilde{f}$ on enhanced elliptic curves is homogeneous of degree $k \in \mathbb{Z}$ if

$$\widetilde{f}(mp+m\Lambda, \mathbb{C}/m\Lambda) = m^k \widetilde{f}(p+\Lambda, \mathbb{C}/\Lambda)$$

for all $m \in \mathbb{C}^\times$.

The theorem which follows will establish an equivalence between enhanced elliptic curves and certain automorphic forms. Before this can be established, we need the following lemma.

**Lemma 4.17.** *Let $\Lambda_\Omega = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ be a lattice, and let $p+\Lambda_\Omega$ be a point of order $N$ on $\mathbb{C}/\Lambda_\Omega$. There exist $c_0, d_0 \in \mathbb{Z}$ with $(c_0, d_0) = 1$ such that*

$$p \equiv \frac{c_0\omega_1 + d_0\omega_2}{N} \pmod{\Lambda}.$$

*Proof.* Since $p$ is a point of order $N$ on $\Lambda_\Omega$ there exist $c, d \in \mathbb{Z}$ such that

$$p = \frac{c\omega_1 + d\omega_2}{N},$$

with $\gcd(c, d, N) = 1$. Bezout's theorem then implies that there exist $a, b, k \in \mathbb{Z}$ such that

$$ad - bc + kN = 1.$$

Reducing this expression modulo $N$ shows that $ad - bc \equiv 1 \pmod{N}$. Thus, writing $\overline{n}$ for the residue class modulo $N$ of an integer $n$, we have that

$$\alpha := \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Because the standard reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective this means there exists a matrix

$$\alpha_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

such that $\overline{\alpha}_0 = \alpha$. In particular, there exist $s, t \in \mathbb{Z}$ such that $c = c_0 + sN$ and $d = d_0 + tN$. We have that

$$p \equiv \frac{(c_0 + sN)\omega_1 + (d_0 + tN)\omega_2}{N} \equiv \frac{c_0\omega_1 + d_0\omega_2}{N} \pmod{\Lambda_\Omega}.$$

Since $\alpha_0 \in \mathrm{SL}_2(\mathbb{Z})$ this means that $(c_0, d_0) = 1$, as required. □

**Theorem 4.18.** *Let $\mathcal{F}_k$ be the set of homogeneous functions $\widetilde{f}$ of degree $k$ on enhanced elliptic curves, which satisfy the following properties*

*(1) The function $\widetilde{f}\left(\frac{1}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$, viewed as a function of $\tau$, is meromorphic on $\mathbb{H}$,*

*(2) For each relatively prime $c, d \in \mathbb{Z}$ the function $\widetilde{f}\left(\frac{c\tau+d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$, viewed as a function of $\tau$, is meromorphic at infinity.*

*Then $\mathcal{F}_{-k}$ is in bijection with $\mathcal{A}_k(\Gamma_1(N))$, the set of automorphic forms of weight $k$ on $\Gamma_1(N)$.*

*Proof.* Let $f \in \mathcal{A}_k(\Gamma_1(N))$ and let $F$ be the homogeneous automorphic form corresponding to $f$. Define a function on enhanced elliptic curves as follows. Let $(p + \Lambda, \mathbb{C}/\Lambda)$ be an enhanced elliptic curve with $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. By Lemma 4.17 we may write

$$p + \Lambda = \frac{c_0\omega_1 + d_0\omega_2}{N} + \Lambda,$$

where $(c_0, d_0) = 1$. Thus, there exist $a, b \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a & b \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, so by Theorem 4.13 it follows that $\Lambda = (a\omega_1 + b\omega_2)\mathbb{Z} \oplus (pN)\mathbb{Z}$ since

$$\begin{pmatrix} a\omega_1 + b\omega_2 \\ pN \end{pmatrix} = \begin{pmatrix} a & b \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Letting $\omega := a\omega_1 + b\omega_2$, define the following function on enhanced elliptic curves:

$$\widetilde{f}(p + \Lambda, \mathbb{C}/\Lambda) := F\begin{pmatrix} \omega \\ pN \end{pmatrix} \tag{4.18}$$

To verify that this is well-defined it must be checked that $F$ is independent of the chosen $\omega$ and that $F$ is independent of the representative $p$. Assume that $p + \Lambda =$

$p' + \Lambda$. Choose $\omega' \in \mathbb{C}$ via the same process as above so that $\Lambda = \omega'\mathbb{Z} \oplus (p'N)\mathbb{Z}$. Since $\Lambda = \omega\mathbb{Z} \oplus (pN)\mathbb{Z}$ there exist $a, b \in \mathbb{Z}$ such that

$$\omega' = a\omega + bpN.$$

Moreover, since $p' - p \in \Lambda$ there exist $c, d \in \mathbb{Z}$ such that $p' = p + c\omega + dpN$. This implies

$$p'N = cN\omega + (dN + 1)pN.$$

Thus, we have that

$$\begin{pmatrix} \omega' \\ p'N \end{pmatrix} = \begin{pmatrix} a & b \\ cN & dN+1 \end{pmatrix} \begin{pmatrix} \omega \\ pN \end{pmatrix}.$$

Since $\{\omega', p'N\}$ and $\{\omega, pN\}$ are both bases for $\Lambda$ and $\gamma := \left(\begin{smallmatrix} a & b \\ cN & dN+1 \end{smallmatrix}\right)$ has integer entries, the argument used in the second half of Theorem 4.13 implies that $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Setting $\det(\gamma) = 1$ immediately shows that $a \equiv 1 \pmod N$ so $\gamma \in \Gamma_1(N)$. Thus

$$\begin{aligned} \widetilde{f}(p' + \Lambda, \mathbb{C}/\Lambda) &= F\begin{pmatrix} \omega' \\ p'N \end{pmatrix} \\ &= F\left(\alpha \begin{pmatrix} \omega \\ pN \end{pmatrix}\right) \\ &= F\begin{pmatrix} \omega \\ pN \end{pmatrix} \\ &= \widetilde{f}(p + \Lambda, \mathbb{C}/\Lambda), \end{aligned}$$

proving that $\widetilde{f}(p' + \Lambda, \mathbb{C}/\Lambda)$ is independent of both the choice of representatives $p$ and basis elements $\omega$. Thus the definition in (4.18) is well defined. Moreover, it is clear from (4.18) and the fact that $F$ is homogeneous of degree $-k$ that $\widetilde{f}(p + \Lambda, \mathbb{C}/\Lambda)$ is homogeneous of degree $-k$.

It remains to check that the function $\widetilde{f}\left(\frac{1}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$ is meromorphic on $\mathbb{H}$ and that $\widetilde{f}\left(\frac{c\tau+d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$ is meromorphic at infinity for all $c, d \in \mathbb{Z}$ with $(c, d) = 1$.

The former is immediate since

$$\widetilde{f}\left(\frac{1}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right) = F\begin{pmatrix}\tau \\ 1\end{pmatrix}$$

and $F\left(\begin{smallmatrix}\tau \\ 1\end{smallmatrix}\right)$ is meromorphic on $\mathbb{H}$. For the latter, let $\tau \in \mathbb{H}$, let $c, d \in \mathbb{Z}$ such that $(c, d) = 1$, and let $f$ be the automorphic form corresponding to the homogeneous automorphic form $F$. Then there exist $a, b \in \mathbb{Z}$ such that $\alpha = \left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by Bézout's identity. Moreover, we have that $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z} = (a\tau + b)\mathbb{Z} \oplus (c\tau + d)\mathbb{Z}$. Thus

$$\begin{aligned}\widetilde{f}\left(\frac{c\tau + d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right) &= F\begin{pmatrix}a\tau + b \\ c\tau + d\end{pmatrix} \\ &= (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= f|_{[\alpha]_k}(\tau).\end{aligned} \tag{4.19}$$

Because $f$ is an automorphic form, this means $\widetilde{f}\left(\frac{c\tau+d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$ is meromorphic at infinity. Thus, $\widetilde{f} \in \mathcal{F}_{-k}$. This establishes that automorphic forms of weight $k$ on $\Gamma_1(N)$ give rise to functions in $\mathcal{F}_{-k}$.

Conversely, given a function $\widetilde{f} \in \mathcal{F}_{-k}$, define

$$F\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix} := \widetilde{f}(\omega_2/N + \Lambda, \mathbb{C}/\Lambda) \tag{4.20}$$

where $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. The fact that $F$ is homogeneous of degree $-k$ is immediate from the fact that $\widetilde{f}$ is homogeneous of degree $-k$. Moreover, let $\gamma = \left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_1(N)$ and let $\Lambda' = (a\omega_1 + b\omega_2)\mathbb{Z} \oplus (c\omega_1 + d\omega_2)\mathbb{Z}$. Then $\Lambda' = \Lambda$ by Theorem 4.13 since

$\gamma \in \mathrm{SL}_2(\mathbb{Z})$. It follows that

$$\begin{aligned} F\left(\gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right) &= F\begin{pmatrix} a\omega_1 + b\omega_2 \\ c\omega_1 + d\omega_2 \end{pmatrix} \\ &= \widetilde{f}\left(\frac{c\omega_1 + d\omega_2}{N} + \Lambda', \mathbb{C}/\Lambda'\right) \\ &= \widetilde{f}\left(\frac{c\omega_1 + d\omega_2}{N} + \Lambda, \mathbb{C}/\Lambda\right) \\ &= \widetilde{f}\left(\frac{\omega_2}{N} + \Lambda, \mathbb{C}/\Lambda\right) \\ &= F\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \end{aligned}$$

where the second to last step holds since $c \equiv 0 \pmod N$ and $d \equiv 1 \pmod N$, so $\frac{c\omega_1+d\omega_2}{N} + \Lambda = \frac{\omega_2}{N} + \Lambda$. This shows that $F$ satisfies (4.13). Moreover, $F\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)$ is meromorphic on $\mathbb{H}$ since $\widetilde{f} \in \mathcal{F}_{-k}$. Again, letting $f$ denote the automorphic form on $\mathbb{H}$ corresponding to $F$, we have $f|_{[\alpha]_k}(\tau) = \widetilde{f}\left(\frac{c\tau+d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right)$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, by reversing the steps in (4.19). Since $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ this means $c, d \in \mathbb{Z}$ are relatively prime. Thus, $f|_{[\alpha]_k}(\tau)$ is meromorphic at infinity since $\widetilde{f} \in \mathcal{F}_{-k}$. This establishes that functions $\widetilde{f} \in \mathcal{F}_{-k}$ give rise to automorphic forms of weight $k$ on $\Gamma_1(N)$. Thus, the equivalence is proved. □

This theorem establishes how automorphic forms for $\Gamma_1(N)$ can be viewed as functions on enhanced elliptic curves. These automorphic forms can also be viewed as families of automorphic forms on $\Gamma(N)$, as will now be described.

In all that follows let $(p_1, p_2) \in \frac{1}{N}\mathbb{Z}^2$ satisfying the property that $(p_1 M, p_2 M) \notin \mathbb{Z}^2$ if $M < N$. Denote elements of $W \subseteq \mathbb{C}^2$ as $\Omega = (\omega_1, \omega_2)^t$.

**Theorem 4.19.** *Let $\{F_{(p_1,p_2)}\}$ be a family of automorphic forms of weight $k$ on $\Gamma(N)$, indexed by $(p_1, p_2)$ as described above. Furthermore, let each form in the family satisfy the properties*

*(Fr1)* $F_{(p_1,p_2)}(\alpha\Omega) = F_{(p_1,p_2)\alpha}(\Omega)$ *for all* $\alpha \in SL_2(\mathbb{Z})$*, where* $(p_1, p_2)\alpha$ *is the usual product of matrices.*

*(Fr2)* *If* $(p_1, p_2) \equiv (q_1, q_2) \pmod{\mathbb{Z}^2}$ *then* $F_{(p_1,p_2)} = F_{(q_1,q_2)}$.

*The set of all such families* $\{F_{(p_1,p_2)}\}$ *is in bijection with* $\mathcal{A}_k(\Gamma_1(N))$.

*Proof.* Because of Theorem 4.18, it suffices to establish a bijection between the families $\{F_{(p_1,p_2)}\}$ and functions in $\mathcal{F}_{-k}$.

Let $\widetilde{f} \in \mathcal{F}_{-k}$. Define a family of forms on $\Gamma(N)$ by defining for each $(p_1, p_2)$ the function

$$F_{(p_1,p_2)}(\Omega) := \widetilde{f}(p_1\omega_1 + p_2\omega_2 + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega),$$

which is homogeneous of degree $-k$. Recall that, by definition, $(p_1, p_2) \in \frac{1}{N}\mathbb{Z}^2$ and $(p_1 M, p_2 M) \notin \mathbb{Z}^2$ when $M < N$. Thus, the point $p_1\omega_1 + p_2\omega_2 + \Lambda_\Omega$ is a point of order $N$ on $\mathbb{C}/\Lambda_\Omega$, as required. It must be shown that each of the functions $F_{(p_1,p_2)}$ is invariant under $\Gamma(N)$, and that they satisfy (Fr1) and (Fr2) along with all meromorphicity conditions. To verify that $F_{(p_1,p_2)}$ is invariant under $\Gamma(N)$ let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. Then

$$F_{(p_1,p_2)}(\gamma\Omega) = \widetilde{f}\big(p_1(a\omega_1 + b\omega_2) + p_2(c\omega_1 + d\omega_2) + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega\big), \tag{4.21}$$

where Theorem 4.13 has again been used to conclude that $\Lambda_{\gamma\Omega} = \Lambda_\Omega$. Now, note that

$$p_1(a\omega_1 + b\omega_2) + p_2(c\omega_1 + d\omega_2) \equiv p_1\omega_1 + p_2\omega_2 \pmod{\Lambda_\Omega}$$

since $a \equiv d \equiv 1 \pmod{N}$ and $b \equiv c \equiv 0 \pmod{N}$. This along with (4.21) implies that (4.13) is satisfied. Property (Fr1) is immediate by letting $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and comparing the definition of $F_{(p_1,p_2)}(\alpha\Omega)$ and $F_{(p_1,p_2)\alpha}(\Omega)$. Property (Fr2) is similarly straightforward to verify by noting that $(p_1, p_2) \equiv (q_1, q_2) \pmod{\mathbb{Z}^2}$ implies

$$p_1\omega_1 + p_2\omega_2 \equiv q_1\omega_1 + q_2\omega_2 \pmod{\Lambda_\Omega}.$$

It remains to show that $f(\tau) := F_{(p_1,p_2)}\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)$ is meromorphic and that $f|_{[\alpha]_k}(\tau)$ is meromorphic at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. The latter condition is clear since

$$\begin{aligned} f|_{[\alpha]_k}(\tau) &= F_{(p_1,p_2)}\begin{pmatrix} \alpha\tau \\ 1 \end{pmatrix} \\ &= \widetilde{f}\big(p_1(\alpha\tau) + p_2 + \Lambda_{\alpha\tau}, \mathbb{C}/\Lambda_{\alpha\tau}\big). \end{aligned}$$

By Lemma 4.17 we may choose $q_1, q_2 \in \mathbb{Z}$ relatively prime such that

$$p_1(\alpha\tau) + p_2 + \Lambda_{\alpha\tau} = \frac{q_1(\alpha\tau) + q_2}{N} + \Lambda_{\alpha\tau}.$$

Thus, $f|_{[\alpha]_k}(\tau)$ is meromorphic at infinity since $\widetilde{f}\left(\frac{q_1(\alpha\tau)+q_2}{N} + \Lambda_{\alpha\tau}, \mathbb{C}/\Lambda_{\alpha\tau}\right)$ is.

To show that $F_{(p_1,p_2)}\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$ is meromorphic, redefine $q_1, q_2 \in \mathbb{Z}$ so that

$$p_1\tau + p_2 + \Lambda_\tau = \frac{q_1\tau + q_2}{N} + \Lambda_\tau,$$

with $(q_1, q_2) = 1$. There exist $a, b \in \mathbb{Z}$ such that $\alpha_p := \left(\begin{smallmatrix}a & b\\ q_1 & q_2\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and we have

$$(q_1\tau + q_2)^{-1}\Lambda_\tau = \Lambda_{\alpha_p\tau}.$$

Thus, recalling that $\widetilde{f}$ is homogeneous of degree $-k$ it follows that

$$\begin{aligned} F_{(p_1,p_2)}\begin{pmatrix}\tau\\1\end{pmatrix} &= \widetilde{f}\left(\frac{q_1\tau + q_2}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right) \\ &= (q_1\tau + q_2)^{-k}\widetilde{f}\left(\frac{1}{N} + \Lambda_{\alpha_p\tau}, \mathbb{C}/\Lambda_{\alpha_p\tau}\right). \end{aligned}$$

This is meromorphic by assumption, so $F_{(p_1,p_2)}\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$ is as well. This establishes that functions $\widetilde{f} \in \mathcal{F}_{-k}$ give rise to automorphic forms of weight $k$ on $\Gamma_1(N)$.

Conversely, let $\{F_{(p_1,p_2)}\}$ be a family of automorphic forms on $\Gamma(N)$ satisfying (Fr1). Define a corresponding function on enhanced elliptic curves as follows. Let $(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega)$ be an enhanced elliptic curve with $\Lambda_\Omega = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Since $p + \Lambda_\Omega$ is a point of order $N$ on $\mathbb{C}/\Lambda_\Omega$ there exists a unique pair of numbers $(p_1, p_2) \in \frac{1}{N}\mathbb{Z}^2$ such that

$$p = (p_1, p_2)\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}. \tag{4.22}$$

Moreover, the pair $(p_1, p_2)$ has the property that $(p_1M, p_2M) \notin \mathbb{Z}^2$ for $M < N$. Define

$$\widetilde{f}(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega) := F_{(p_1,p_2)}(\Omega).$$

To verify that this is well defined, it must be shown that $\widetilde{f}(p+\Lambda_\Omega, \mathbb{C}/\Lambda_\Omega)$ is independent of the choice of representative $p$, and also that it is independent of the choice of basis for $\Lambda_\Omega$. The fact that $\widetilde{f}$ is independent of the representative $p$ follows directly from (Fr2). Indeed, if $q = q_1\omega_1 + q_2\omega_2$ is another point such that $p + \Lambda_\Omega = q + \Lambda_\Omega$ it follows that $(p_1, p_2) \equiv (q_1, q_2) \pmod{\mathbb{Z}^2}$, so

$$\widetilde{f}(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega) = \widetilde{f}(q + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega).$$

To see that $\widetilde{f}$ is independent of the chosen basis, let $\Omega' = (\omega_1', \omega_2')^t$ where $\Lambda_{\Omega'} = \Lambda_\Omega$ and let $(p_1', p_2') \in \frac{1}{N}\mathbb{Z}^2$ be the unique pair such that

$$p = (p_1', p_2')\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix}. \tag{4.23}$$

By Theorem 4.13 we have $\Omega = \alpha\Omega'$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. So, it follows from (4.22)

$$p = (p_1, p_2)\Omega = (p_1, p_2)\alpha\Omega'.$$

Since the pair $(p_1', p_2')$ in (4.23) is unique it follows that $(p_1', p_2') = (p_1, p_2)\alpha$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Thus,

$$\begin{aligned} F_{(p_1', p_2')}(\Omega') &= F_{(p_1,p_2)\alpha}(\Omega') \\ &= F_{(p_1,p_2)}(\alpha\Omega') \\ &= F_{(p_1,p_2)}(\Omega), \end{aligned}$$

where (Fr1) was used from the first to the second line. This establishes that $\widetilde{f}$ is well defined.

To see that $\widetilde{f}$ is homogeneous of degree $-k$, let $m \in \mathbb{C}^\times$. Note that if

$$\widetilde{f}(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega) = F_{(p_1,p_2)}(\Omega)$$

then

$$\widetilde{f}(mp + m\Lambda_\Omega, \mathbb{C}/m\Lambda_\Omega) = F_{(p_1,p_2)}(m\Omega)$$

since $mp = (p_1, p_2)(m\Omega)$. This makes it clear that $\widetilde{f}$ is homogeneous of degree $-k$

because

$$\widetilde{f}(mp + m\Lambda_\Omega, \mathbb{C}/m\Lambda_\Omega) = F_{(p_1,p_2)}(m\Omega)$$
$$= m^{-k}F_{(p_1,p_2)}(\Omega)$$
$$= m^{-k}f(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega).$$

All that remains is to verify the meromorphicity conditions. These are straightforward, however, because

$$\widetilde{f}\left(\frac{1}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right) = F_{\left(0,\frac{1}{N}\right)}(\Omega)$$

is meromorphic since $F$ is an automorphic form. Moreover,

$$\widetilde{f}\left(\frac{c\tau + d}{N} + \Lambda_\tau, \mathbb{C}/\Lambda_\tau\right) = F_{\left(\frac{c}{N},\frac{d}{N}\right)}(\Omega)$$

is meromorphic at infinity since $F$ is an automorphic form. □

We close the section by giving the definition of a special family of forms on $\Gamma(N)$. First, recall that given $\zeta_N$ a primitive $N$-th root of unity, and $d \in \mathbb{Z}$ relatively prime to $N$ there is a unique automorphism $\sigma_d : \mathbb{Q}(\zeta_N) \to \mathbb{Q}(\zeta_N)$ which sends $\zeta_N \mapsto \zeta_N^d$. This automorphism extends naturally to the field of formal Laurent series $\mathbb{Q}(\zeta_n)((q_\tau^{1/N}))$ by acting on the coefficients. In what follows, again let the indices $(p_1, p_2) \in (1/N)\mathbb{Z}^2$ be such that $(p_1 M, p_2 M) \notin \mathbb{Z}^2$ if $M < N$.

**Definition 4.20.** Let $\{F_{(p_1,p_2)}\}$ be a family of automorphic forms of weight $k$ for $\Gamma(N)$. Let $\sigma_d$ be the automorphism on $\mathbb{Q}(\zeta_N)((q_\tau^{1/N}))$ described directly above. The family of forms $\{F_{(p_1,p_2)}\}$ is called a *Fricke family* of weight $k$ and level $N$ if each $F_{(p_1,p_2)}$ satisfies the following three properties.

(Fr1) $F_{(p_1,p_2)}(\alpha\Omega) = F_{(p_1,p_2)\alpha}(\Omega)$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, where $(p_1, p_2)\alpha$ is the usual product of matrices.

(Fr2) If $(p_1, p_2) \equiv (q_1, q_2) \pmod{\mathbb{Z}^2}$ then $F_{(p_1,p_2)} = F_{(q_1,q_2)}$.

(Fr3) Given $\tau \in \mathbb{H}$, the coefficients $a_n$ in the Fourier expansion

$$F_{(p_1,p_2)}\begin{pmatrix}\tau\\1\end{pmatrix} = \sum_{n=n_0}^{\infty} a_n q_\tau^{n/N}$$

all lie in the field $\mathbb{Q}(\zeta_N)$.

(Fr4) $F_{(p_1,p_2)}^{\sigma_d} = F_{(p_1,dp_2)}$ for all $d$ relatively prime to $N$.

In light of Theorems 4.18 and 4.19 a Fricke family can be thought of as a particular type of automorphic form on $\Gamma_1(N)$, or as a particular type of homogeneous function on enhanced elliptic curves. In the following theorem, we take the latter approach, and explain an important property of the Fricke families.

**Theorem 4.21.** *Let $\{F_{(p_1,p_2)}(\Omega)\}$ be a Fricke family of weight $k$ and level $N$ with $\Omega = (\omega_1, \omega_2)^t$ and $\omega_1/\omega_2 \in \mathbb{H}$. Furthermore, let $p = p_1\omega_1 + p_2\omega_2$ be a point of order $N$ on $\mathbb{C}/\Lambda_\Omega$, and let $f \in \mathcal{F}_{-k}$ be the homogeneous function on enhanced elliptic curves which corresponds to the Fricke family. Then*

$$f(\overline{p} + \Lambda_{\overline{\Omega}}, \mathbb{C}/\Lambda_{\overline{\Omega}}) = \overline{f(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega)},$$

*where the bar denotes complex conjugation.*

*Proof.* This follows from writing down the $q_\tau^{1/N}$ expansion of the inhomogeneous automorphic form on $\Gamma(N)$ which corresponds to $f(p + \Lambda_\Omega, \mathbb{C}/\Lambda_\Omega)$. Then, one takes the complex conjugate and uses (FR4) with $d = -1$. The details are found in Proposition 1.4 of [10, Ch. 2]. □

## 4.5 Klein Forms and the Siegel Functions

Once again, let $\Omega = (\omega_1, \omega_2)^t$ be a column vector of complex numbers such that $\omega_1/\omega_2 \in \mathbb{H}$. Let $t = (t_1, t_2) \in \mathbb{R}^2$. Note that

$$t\Omega = (t_1, t_2)\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix} = t_1\omega_1 + t_2\omega_2$$

Define the Klein Forms

$$K_t(\Omega) = e^{-\eta(t\Omega,\Lambda_\Omega)(t\Omega)/2}\sigma(t\Omega, \Lambda_\Omega). \tag{4.24}$$

At times $K_t(\Omega)$ will be written as $K(t\Omega, \Lambda_\Omega)$, following the format used for the $\eta$ and $\sigma$ functions.

**Theorem 4.22.** *The Klein forms satisfy the following properties:*

*(K1)* $K_t(m\Omega) = mK_t(\Omega)$ *for all* $m \in \mathbb{C}^\times$.

*(K2)* $K_t(\alpha\Omega) = K_{t\alpha}(\Omega)$ *for all* $\alpha \in SL_2(\mathbb{Z})$, *where* $t\alpha$ *is a product of a row vector and a matrix.*

*(K3)* $K_{t+u}(\Omega) = \varepsilon(t, u)K_t(\Omega)$ *for any* $u = (u_1, u_2) \in \mathbb{Z}^2$, *where*

$$\varepsilon(t, u) = -(-1)^{(u_1+1)(u_2+1)}e^{-\pi i(u_1t_2-u_2t_1)}.$$

*(K4) If* $t \in \frac{1}{N}\mathbb{Z}^2$ *then* $K_t(\gamma\Omega) = \varepsilon_t(\gamma)K_t(\Omega)$ *for all* $\gamma \in \Gamma(N)$, *where* $\varepsilon_t(\gamma)$ *is a* $(2N)$*-th root of unity.*

*Proof.* Property (K1) follows immediately from (4.24) along with (4.2) and (4.5), which show that $\eta$ is homogeneous of degree $-1$ and $\sigma$ is homogeneous of degree 1. To verify property (K2), let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Then $\Lambda_{\alpha\Omega} = \Lambda_\Omega$ so

$$K_t(\alpha\Omega) = K(t(\alpha\Omega), \Lambda_\Omega) = K((t\alpha)\Omega, \Lambda_\Omega) = K_{t\alpha}(\Omega).$$

For (K3) let $u = (u_1, u_2) \in \mathbb{Z}^2$. To simplify notation, let $z = t\Omega$ and $\omega = u\Omega$ and note that $\omega \in \Lambda_\Omega$ since $u \in \mathbb{Z}^2$. Then

$$K_{t+u}(\Omega) = e^{-\eta(z+\omega,\Lambda_\Omega)(z+\omega)/2}\sigma(z + \omega, \Lambda_\Omega).$$

Applying the functional equation for $\sigma$ from Theorem 4.6 produces

$$K_{t+u}(\Omega) = \psi(\omega)e^{-[\omega\eta(z)-z\eta(\omega)]/2}K_t(\Omega). \tag{4.25}$$

Finally, by expanding $z = t\Omega$ and $\omega = u\Omega$ and simplifying, we may rewrite the exponent as follows:

$$\begin{aligned}\omega\eta(z) - z\eta(\omega) &= t_2u_1(\omega_1\eta_2 - \omega_2\eta_1) + t_1u_2(\omega_2\eta_1 - \omega_1\eta_2)\\ &= 2\pi i(t_2u_1 - t_1u_2),\end{aligned}$$

where the last equality follows from the Legendre relation of Theorem 4.5. Substituting this expression in to (4.25) and using Corollary 4.7 to rewrite $\psi(\omega)$ gives the desired result

$$K_{t+u}(\Omega) = -(-1)^{(u_1+1)(u_2+1)}e^{-\pi i(t_2u_1-t_1u_2)}K_t(\Omega),$$

which proves (K3).

To prove (K4), let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(N)$ and let $t = (t_1, t_2) \in \frac{1}{N}\mathbb{Z}^2$. To express $K_t(\gamma\Omega) = K_{t\gamma}(\Omega)$ in terms of $K_t(\Omega)$ we will rewrite $t\gamma$ as a sum of $t$ and another row vector with integer entries, and then use (K3). First note that

$$\begin{aligned}(t_1, t_2)\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= (at_1 + ct_2, bt_1 + dt_2)\\ &= \big(t_1 + (a-1)t_1 + ct_2, t_2 + bt_1 + (d-1)t_2\big) \qquad (4.26)\\ &= (t_1, t_2) + \big((a-1)t_1 + ct_2, bt_1 + (d-1)t_2\big)\\ &= (t_1, t_2) + (u_1, u_2),\end{aligned}$$

where the pair $(u_1, u_2)$ is defined by the second to last line. Recall that $(t_1, t_2) \in \frac{1}{N}\mathbb{Z}^2$ but since $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(N)$ it follows that $(u_1, u_2) \in \mathbb{Z}^2$. Thus, (K3) implies

$$K_{t\gamma}(\Omega) = -(-1)^{(u_1+1)(u_2+1)}e^{-\pi i(t_2u_1-t_1u_2)}K_t(\Omega).$$

It is clear that the factor in front of $K_t(\Omega)$ is a $(2N)$-th root of unity. Indeed, $\big(-(-1)^{(u_1+1)(u_2+1)}\big)^2 = 1$ and letting $t = \big(\frac{r_1}{N}, \frac{r_2}{N}\big)$ produces

$$t_2u_1 - t_1u_2 = \frac{r_1u_1 - r_2u_2}{N} \in \frac{1}{N}\mathbb{Z},$$

so $\big(e^{-\pi i(t_2u_1-t_1u_2)}\big)^{2N} = 1$. This completes the proof of (K4).

□

The homogeneous functions $K_t(\Omega)$ define functions on the upper half plane

$$K_t(\tau) := K_t \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Using the product expansion (4.11) for the $\sigma$ function along with the Legendre relation from Theorem 4.5 produces a useful product expansion for these functions. Again, let $t = (t_1, t_2)$ and

$$z = (t_1, t_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = t_1\tau + t_2.$$

Note that in this case, the Legendre relation reads

$$\tau\eta(1, \Lambda_\tau) - 1 \cdot \eta(\tau, \Lambda_\tau) = \tau\eta_2 - \eta_1 = 2\pi i.$$

Substituting the product expansion for $\sigma$ into the definition of the Klein functions in (4.24) produces

$$K_t(\tau) = \frac{1}{2\pi i} e^{\frac{1}{2}(\eta_2 z^2 - \eta(z,\Lambda_\tau)z)} (q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty} \frac{(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1})}{(1 - q_\tau^n)^2}.$$

Using linearity of the $\eta$ function along with the Legendre relation allows the exponent to be simplified as

$$\begin{aligned} \frac{1}{2}(z^2\eta_2 - \eta(z, \Lambda_\tau)z) &= \frac{z}{2}(z\eta_2 - t_1\eta_1 - t_2\eta_2) \\ &= \frac{z}{2}(t_1\tau\eta_2 + t_2\eta_2 - t_1\eta_1 - t_2\eta_2) \\ &= \frac{t_1 z}{2}(\tau\eta_2 - \eta_1) \\ &= \pi i t_1 z. \end{aligned}$$

This produces $e^{\pi i t_1 z} = e^{\pi i t_1 t_2} q_\tau^{t_1^2/2}$. So the following product expansion holds:

$$K_t(\tau) = \frac{1}{2\pi i} e^{\pi i t_1 t_2} q_\tau^{t_1^2/2} (q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty} \frac{(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1})}{(1 - q_\tau^n)^2}. \tag{4.27}$$

**Corollary 4.23.** *Let $t = (t_1, t_2) \in \frac{1}{N}\mathbb{Z}^2$ be such that $t \notin \mathbb{Z}^2$. The functions $K_t(\tau)^{2N}$ are weakly modular of weight $-2N$ on $\Gamma(N)$. Furthermore, they are holomorphic and* *non-vanishing on* $\mathbb{H}$.

*Proof.* The definition of the Klein functions in (4.24) shows that $K_t(\tau) = 0$ if and only if $\sigma(t_1\tau + t_2, \Lambda_\tau) = 0$. The latter function has zeros precisely when $t_1\tau + t_2 \in \Lambda_\tau$, and this occurs only when $(t_1, t_2) \in \mathbb{Z}^2$. Thus, $K_t(\tau)^{2N}$ is non-vanishing. The expansion (4.27) shows that $K_t(\tau)$ is holomorphic. Since $K_t(\Omega)$ is homogeneous of degree 1 by (K1), it follows that $K_t(\Omega)^{2N}$ is homogeneous of degree $2N$. Moreover, from (K4) if $\gamma \in \Gamma(N)$ then

$$K_t(\gamma\Omega)^{2N} = \varepsilon_t(\gamma)^{2N} K_t(\Omega)^{2N} = K_t(\Omega)^{2N}.$$

These properties together show that $K_t(\tau)$ is weakly modular of weight $-2N$ for $\Gamma(N)$. □

## 4.6 The Siegel Functions

For all that follows, fix the twelfth root of the $\Delta$ function

$$\Delta^{1/12}(\tau) := 2\pi i \eta^2(\tau),$$

where $\eta$ is the Dedekind $\eta$ function defined in (4.16). The expression on the right is indeed a twelfth root of $\Delta$, as seen from (4.17). Given $\tau \in \mathbb{H}$ let $z = t_1\tau + t_2$ and $t = (t_1, t_2)$. Define the Siegel functions

$$g_t(\tau) := K_t(\tau)\Delta^{1/12}(\tau).$$

A product expansion for $g_t(\tau)$ can be deduced directly from (4.16) and (4.27):

$$g_t(\tau) = e^{\pi i t_1 t_2} q_\tau^{t_1^2/2} q_\tau^{1/12} (q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}).$$

Furthermore, if $t_2 \in \frac{1}{N}\mathbb{Z}$ then

$$\begin{aligned} q_z^{1/2} - q_z^{-1/2} &= q_z^{-1/2}(q_z - 1) \\ &= e^{-\pi i t_2} q_\tau^{-t_1/2}(q_z - 1), \end{aligned}$$

and substituting into the product formula above produces

$$g_t(\tau) = -e^{\pi i t_2(t_1-1)} q_\tau^{\frac{1}{2}(t_1^2 - t_1 + \frac{1}{6})}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}), \tag{4.28}$$

which one notes is the same as the definition of the Siegel functions at the start of Section 3.1.

**Theorem 4.24.** *Let $t \in \frac{1}{N}\mathbb{Z}^2$ be such that $t \notin \mathbb{Z}^2$. Then the function $g_t(\tau)^{12N}$ is an automorphic form of weight 0 on $\Gamma(N)$, and it is holomorphic and non-vanishing on $\mathbb{H}$. Moreover, $g_t(\tau)^{12N} \in \mathbb{Q}(\zeta_N)((q_\tau^{1/N}))$, where $\zeta_N$ is a primitive $N$-th root of unity.*

*Proof.* The discriminant function $\Delta(\tau)$ is non-vanishing, and from Corollary 4.23, so is the function $K_t(\tau)$. This means that $g_t(\tau)$ is non-vanishing as well. Moreover, since $K_t(\tau)$ and $\Delta(\tau)$ are both holomorphic, this means $g_t(\tau)^{12} = K_t(\tau)^{12}\Delta(\tau)$ is also holomorphic. Next, if $\gamma \in \Gamma(N)$ then

$$\begin{aligned} g_t(\gamma\tau)^{12N} &= K_t(\gamma\tau)^{12N}\Delta(\gamma\tau)^N \\ &= (c\tau+d)^{-12N}K_t(\tau)^{12N}\cdot(c\tau+d)^{12N}\Delta(\tau) \\ &= g_t(\tau)^{12N}, \end{aligned}$$

since $K_t(\tau)^{12N}$ is weight $-12N$ invariant for $\Gamma(N)$ and $\Delta(\tau)$ is weight 12 invariant for $\mathrm{SL}_2(\mathbb{Z})$. Thus, $g_t(\tau)^{12N}$ is weakly modular of weight 0 for $\Gamma(N)$. All that remains is to show that $g_t(\alpha\tau)^{12N}$ is meromorphic at infinity for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, and that the Fourier expansion of $g_t(\tau)^{12N}$ has coefficients in $\mathbb{Q}(\zeta_N)$. Given $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ it follows that

$$\begin{aligned} g_t(\alpha\tau)^{12N} &= K_t^{12N}\begin{pmatrix}\alpha\tau \\ 1\end{pmatrix}\Delta(\alpha\tau)^N \\ &= (c\tau+d)^{-12N}K_t^{12N}\left(\alpha\begin{pmatrix}\tau \\ 1\end{pmatrix}\right)\cdot(c\tau+d)^{12N}\Delta(\tau)^N \\ &= K_{t\alpha}^{12N}(\tau)\Delta(\tau)^N \\ &= g_{t\alpha}(\tau)^{12N}, \end{aligned} \tag{4.29}$$

where the second line uses the facts that $\Delta(\tau)$ is weight 12 invariant for $\mathrm{SL}_2(\mathbb{Z})$ and that $K_t(\Omega)$ is homogeneous of degree 1. The third line uses (K2). Since $t \in \frac{1}{N}\mathbb{Z}^2$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ then $t\alpha \in \frac{1}{N}\mathbb{Z}^2$. Thus, to prove that $g_t(\alpha\tau)^{12N}$ is meromorphic at infinity, it suffices to prove that $g_t(\tau)^{12N}$ is meromorphic at infinity for all $t \in \frac{1}{N}\mathbb{Z}^2$.

To show this, first define the products

$$P_t(\tau) := \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}),$$

and

$$Q_t(\tau) := e^{\pi i t_2(t_1-1)} q_\tau^{\frac{1}{2}(t_1^2 - t_1 + \frac{1}{6})}(q_z - 1).$$

Since $g_t(\tau) = Q_t(\tau)P_t(\tau)$, it suffices to show that the $Q_t(\tau)^{12N} \in \mathbb{Q}(\zeta_N)[q_\tau^{1/N}]$ and that $P_t(\tau)^{12N} \in \mathbb{Q}(\zeta_N)((q_\tau^{1/N}))$.

First, consider the product $Q_t(\tau)$. Writing $(t_1, t_2) = \left(\frac{r_1}{N}, \frac{r_2}{N}\right)$ and recalling that $q_z = e^{2\pi i t_2} q_\tau^{t_1}$ produces

$$Q_t(\tau) = e^{\pi i r_2(r_1 - N)/N^2} q_\tau^{(6r_1^2 - 6Nr_1 + N^2)/12N^2}(e^{2\pi i r_2/N} q_\tau^{t_1/N} - 1). \tag{4.30}$$

From this expression, it is clear that $Q_t(\tau)^{12N} \in \mathbb{Q}(\zeta_N)[q_\tau^{1/N}]$.

Next, consider the product $P_t(\tau)$. Note that

$$q_\tau^n q_z^{\pm 1} = e^{2\pi i \tau n} e^{\pm 2\pi i(t_1\tau + t_2)} = e^{\pm 2\pi i r_2/N} q_\tau^{\frac{nN \pm r_1}{N}}. \tag{4.31}$$

Thus, expanding the partial products of $P_t(\tau)$ shows that $P_t(\tau) \in \mathbb{Q}(\zeta_N)((q_\tau^{1/N}))$. □

Since the functions $g_t(\tau)^{12N}$ are weakly modular of weight 0, this means that the corresponding homogeneous automorphic forms are defined by

$$g_t^{12N}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} := g_t^{12N}\left(\frac{\omega_1}{\omega_2}\right).$$

As done here, both functions will be denoted by $g_t^{12N}$, and it should be clear from context which function is being referred to.

**Theorem 4.25.** *Let $t = (t_1, t_2) \in \frac{1}{N}\mathbb{Z}^2$ with $t \notin \mathbb{Z}^2$. Then the family of functions $\{g_t^{12N}\}$ form a Fricke family.*

*Proof.* To verify property (Fr1), let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and let $\Omega = (\omega_1, \omega_2)^t$ where $\frac{\omega_1}{\omega_2} \in \mathbb{H}$. Then

$$\begin{aligned} g_t^{12N}(\alpha\Omega) &= g_t^{12N}\left(\alpha \cdot \frac{\omega_1}{\omega_2}\right) \\ &= g_{t\alpha}^{12N}\left(\frac{\omega_1}{\omega_2}\right) \\ &= g_{t\alpha}^{12N}(\Omega), \end{aligned}$$

where the second line follows from (4.29). This shows that (Fr1) holds. To verify property (Fr2), let $t' = (t_1', t_2')$ be a point such that $t' \equiv t \pmod{\mathbb{Z}^2}$. This implies that

$$u := (u_1, u_2) = t' - t \in \mathbb{Z}.$$

Thus, noting that $t' = t + u$ we have

$$\begin{aligned} g_{t'}^{12N}(\tau) &= K_{t+u}^{12N}(\tau)\Delta^N(\tau) \\ &= \varepsilon(t, u)^{12N} K_t^{12N}(\tau)\Delta^N(\tau) \\ &= K_t^{12N}(\tau)\Delta^N(\tau) \\ &= g_t^{12N}(\tau), \end{aligned}$$

where (K3) was used from the first to the second line. This verifies property (Fr2).

Furthermore, property (Fr3) follows from Theorem 4.24, and property (Fr4) follows from (4.30) and (4.31). □

# 5 The Siegel-Ramachandra Invariants

Throughout this chapter let $k$ be a quadratic imaginary number field and $\mathfrak{m} \neq (1)$ an integral ideal of $k$. Furthermore, let $N_\mathfrak{m}$ denote the smallest positive integer in $\mathfrak{m}$, and let $w_\mathfrak{m}$ be the number of roots of unity of $k_\mathfrak{m}$ which are congruent to 1 modulo $\mathfrak{m}$.

## 5.1 Definition and Properties

It will be shown that the Siegel functions of Section 4.6 can be used to define certain class invariants called the Siegel-Ramachandra invariants. In fact, any $f \in \mathcal{A}_0(\Gamma_1(N_\mathfrak{m}))$ can be used to define a function which only depends on $C \in Cl_\mathfrak{m}(k)$. This is where we begin.

Recall from Theorem 4.18 that $\mathcal{A}_0(\Gamma_1(N_\mathfrak{m}))$ is in bijection with $\mathcal{F}_0$, the set of homogeneous functions of degree 0 on enhanced elliptic curves. Given a class $C \in Cl_\mathfrak{m}(k)$ let $\mathfrak{c}$ be an integral ideal in $C$. Define

$$f_\mathfrak{m}(C) := \widetilde{f}(1 + \mathfrak{m}\mathfrak{c}^{-1}, \mathbb{C}/\mathfrak{m}\mathfrak{c}^{-1}),$$

where the function on the right is the function on enhanced elliptic curves arising from $f \in \mathcal{A}_0(\Gamma_1(N_\mathfrak{m}))$. To verify that this definition makes sense, two things must be checked. First, it must be verified that $1 + \mathfrak{m}\mathfrak{c}^{-1}$ is a point of order $N_\mathfrak{m}$ on $\mathbb{C}/\mathfrak{m}\mathfrak{c}^{-1}$. Second, it must be verified that $f_\mathfrak{m}(C)$ is independent of the choice of $\mathfrak{c} \in C$.

The point $1+\mathfrak{m}\mathfrak{c}^{-1}$ is of order $N_\mathfrak{m}$ if and only if $N_\mathfrak{m}$ is the smallest positive integer in $\mathfrak{m}\mathfrak{c}^{-1}$. To see that this is the case, note that since $N_\mathfrak{m} \in \mathfrak{m}$ and $\mathfrak{c}$ is an integral ideal we have

$$N_{\mathfrak{m}} \cdot \mathfrak{c} \subseteq \mathfrak{m}\mathcal{O}_k \subseteq \mathfrak{m}.$$

Thus, $\mathfrak{m} \mid N_{\mathfrak{m}} \cdot \mathfrak{c}$ so there is some integral ideal $\mathfrak{a}$ such that $N_{\mathfrak{m}} \cdot \mathfrak{c} = \mathfrak{a}\mathfrak{m}$. So

$$N_{\mathfrak{m}} = \mathfrak{a}\mathfrak{m}\mathfrak{c}^{-1} \subseteq \mathfrak{m}\mathfrak{c}^{-1}.$$

This proves that $N_{\mathfrak{m}} \in \mathfrak{m}\mathfrak{c}^{-1}$. Moreover, $N_{\mathfrak{m}}$ is in fact the smallest positive integer in $\mathfrak{m}\mathfrak{c}^{-1}$. Indeed, if $M < N_{\mathfrak{m}}$ is a positive integer such that $M \in \mathfrak{m}\mathfrak{c}^{-1}$ then $M \cdot \mathfrak{c} \subseteq \mathfrak{m}$, so $\mathfrak{m} \mid M \cdot \mathfrak{c}$. However, $(\mathfrak{m}, \mathfrak{c}) = 1$ so in fact $\mathfrak{m} \mid (M)$, implying that $(M) \subseteq \mathfrak{m}$. This means $M \in \mathfrak{m}$ which is a contradiction since $N_{\mathfrak{m}}$ is the smallest positive integer in $\mathfrak{m}$.

We now explain why $f_{\mathfrak{m}}(C)$ is independent of the choice of $\mathfrak{c} \in C$. Let $\mathfrak{c}_1$ be another integral ideal in $C$. This means there exists an $\alpha \in R_{\mathfrak{m}}(k)$ such that $\mathfrak{c}_1 = \alpha\mathfrak{c}$, meaning

$$\mathfrak{c}^{-1} = \alpha\mathfrak{c}_1^{-1}.$$

Using this, along with the fact that the function $f$ on enhanced elliptic curves is homogeneous of degree zero, we have

$$\begin{aligned} f(1 + \mathfrak{m}\mathfrak{c}_1^{-1}, \mathbb{C}/\mathfrak{m}\mathfrak{c}_1^{-1}) &= \widetilde{f}(\alpha + \mathfrak{m}\alpha\mathfrak{c}_1^{-1}, \mathbb{C}/\mathfrak{m}\alpha\mathfrak{c}_1^{-1}) \\ &= \widetilde{f}(\alpha + \mathfrak{m}\mathfrak{c}^{-1}, \mathbb{C}/\mathfrak{m}\mathfrak{c}^{-1}). \end{aligned} \tag{5.1}$$

We claim that $\alpha + \mathfrak{m}\mathfrak{c}^{-1} = 1 + \mathfrak{m}\mathfrak{c}^{-1}$. This can be verified as follows. Let $c \in \mathfrak{c}$ and note that $c(\alpha - 1) = c\alpha - c$. However, $\alpha \in \mathfrak{c}^{-1}$ since

$$(\alpha) = \mathfrak{c}_1\mathfrak{c}^{-1} \subseteq \mathfrak{c}^{-1}.$$

This means that $c\alpha \in \mathcal{O}_k$ so $c\alpha - c \in \mathcal{O}_k$ and $\mathfrak{c}(\alpha - 1)$ is integral. Moreover, since $(\mathfrak{m}, \mathfrak{c}) = 1$ and $\alpha \in R_{\mathfrak{m}}(k)$ we have

$$\operatorname{ord}_{\mathfrak{p}}(\mathfrak{c}(\alpha - 1)) = \operatorname{ord}_{\mathfrak{p}}(\alpha - 1) \geq \operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})$$

for all $\mathfrak{p} \mid \mathfrak{m}$. This implies that $\mathfrak{m} \mid \mathfrak{c}(\alpha - 1)$ so there exists an integral ideal $\mathfrak{a}$ such that

$$(\alpha - 1) = \mathfrak{a}\mathfrak{m}\mathfrak{c}^{-1} \subseteq \mathfrak{m}\mathfrak{c}^{-1}.$$

Thus, $1+\mathfrak{m}\mathfrak{c}^{-1} = \alpha + \mathfrak{m}\mathfrak{c}^{-1}$ as claimed. Using this along with (5.1) shows that $f_{\mathfrak{m}}(C)$ is independent of the integral ideal $\mathfrak{c} \in C$.

We now apply this construction using the Siegel functions of Section 4.6. Let $t \in \frac{1}{N_{\mathfrak{m}}}\mathbb{Z}^2$ with $t \notin \mathbb{Z}^2$. Then the family of forms $\{g_t^{12N_{\mathfrak{m}}}\}$ is a Fricke family of weight zero according to Theorem 4.25. By Theorem 4.19 this family gives rise to a single form $g^{12N_{\mathfrak{m}}} \in \mathcal{A}_0(\Gamma_1(N))$. Following the construction above, we define the Siegel-Ramachandra invariant as

$$g_{\mathfrak{m}}(C) := \widetilde{g}(1+\mathfrak{m}\mathfrak{c}^{-1}, \mathbb{C}/\mathfrak{m}\mathfrak{c}^{-1})^{12N_{\mathfrak{m}}}. \tag{5.2}$$

We will now discuss some important properties of the Siegel-Ramachandra invariant, but first, we need the following theorem.

**Theorem 5.1.** *Let $\{F_{p_1,p_2}\}$ be a Fricke family of weight 0 and level $N_{\mathfrak{m}}$ and let $f_{\mathfrak{m}}$ be the corresponding function on $Cl_{\mathfrak{m}}(k)$ as defined above. Furthermore, let $\varphi$ denote the Artin isomorphism. Then $f_{\mathfrak{m}}(C) \in k_{\mathfrak{m}}^{\times}$ and*

$$f_{\mathfrak{m}}(C)^{\varphi(C')} = f_{\mathfrak{m}}(C'C)$$

*for all $C, C' \in Cl_{\mathfrak{m}}(k)$.*

*Proof.* This follows from the Shimura reciprocity law, as explained in the proof of Theorem 1.1 of [10, Ch. 11]. □

**Theorem 5.2.** *The Siegel-Ramachandra invariants $g_{\mathfrak{m}}(C)$ satisfy the following properties.*

*(SR1)* $g_{\mathfrak{m}}(C) \in k_{\mathfrak{m}}^{\times}$.

*(SR2) If $\varphi$ is the Artin isomorphism, then $g_{\mathfrak{m}}(1)^{\varphi(C)} = g_{\mathfrak{m}}(C)$.*

*(SR3) If $\mathfrak{m}$ is divisible by at least two distinct prime ideals then $g_{\mathfrak{m}}(C)$ is a unit. If $\mathfrak{m} = \mathfrak{p}^a$ for some $a \in \mathbb{N}$ then $g_{\mathfrak{m}}(C)$ is an $\{\infty, \mathfrak{p}\}$-unit.*

*(SR4) The algebraic number*

$$\frac{g_{\mathfrak{m}}(C)}{g_{\mathfrak{m}}(C)^{\sigma}}$$

*is a unit for all $\sigma \in G_{\mathfrak{m}}$.*

*(SR5) The extension* $k_{\mathfrak{m}}(g_{\mathfrak{m}}(C)^{1/12N_{\mathfrak{m}}})/k$ *is abelian.*

*(SR6) If* $\chi$ *is a character of* $Cl_{\mathfrak{m}}(k)$ *and* $\mathfrak{f}_\chi$ *denotes the conductor of the extension* $k_{\mathfrak{m}}^{\ker\chi}/k$ *then*

$$N_{k_{\mathfrak{m}}/k_{\mathfrak{f}_\chi}}(\varepsilon_{\mathfrak{m}})^{w_{\mathfrak{f}_\chi}/w_{\mathfrak{m}}} = (\varepsilon_{\mathfrak{f}_\chi}^{\alpha})^{N_{\mathfrak{m}}/N_{\mathfrak{f}_\chi}},$$

*where*

$$\alpha = \prod_{\substack{\mathfrak{p}\mid\mathfrak{m} \\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \left(1 - \sigma_{\mathfrak{p}}^{-1}\right).$$

*Proof.* The Siegel functions $\{g_t^{12N_{\mathfrak{m}}}\}$ indexed by $t \in \frac{1}{N_{\mathfrak{m}}}\mathbb{Z}^2 \setminus \mathbb{Z}^2$ form a Fricke family by Theorem 4.25. Thus, properties (SR1) and (SR2) follow from Theorem 5.1.

For properties (SR3) and (SR4) see [10, Ch. 11] Theorem 1.2 and also [22, Ch. 4 §3]. Property (SR5) is a consequence of Lemma 9 in [19]. Finally, for (SR6) see [10, Ch. 11] Theorem 1.4. □

## 5.2 Stark's Conjecture for Quadratic Imaginary $k$

The Siegel-Ramachandra invariant $g_{\mathfrak{m}}(C)$ is closely related to the special values of the Siegel functions which appear in the expression (3.31) for $L'_{\mathfrak{m}}(0,\psi)$. Before explaining this connection in detail, we need two lemmas.

**Lemma 5.3.** *Let* $\mathfrak{a}$ *be an ideal of* $k$*, and let* $\mathfrak{a} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ *with* $\omega_1/\omega_2 \in \mathbb{H}$*. Then*

$$tr(\omega_2)\omega_1 - tr(\omega_1)\omega_2 = \mathbb{N}(\mathfrak{a})\sqrt{d_k}.$$

*Proof.* Letting $d(\mathfrak{a})$ denote the discriminant of $\mathfrak{a}$ we have

$$d(\mathfrak{a}) = d_k(\mathbb{N}(\mathfrak{a}))^2.$$

On the other hand, we may also write

$$d(\mathfrak{a}) = \det\begin{pmatrix}\omega_1 & \overline{\omega}_1 \\ \omega_2 & \overline{\omega}_2\end{pmatrix}^2 = (\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1)^2.$$

These two expressions together imply $(\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1)^2 = d_k(\mathbb{N}(\mathfrak{a}))^2$. Note that because

$k$ is quadratic imaginary $d_k < 0$. Also, $(\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1)^2 < 0$ because

$$\begin{aligned}\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1 &= \omega_2\overline{\omega}_2\left(\frac{\omega_1}{\omega_2} - \frac{\overline{\omega}_1}{\overline{\omega}_2}\right) \\ &= 2i|\omega_2|^2 \cdot \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right).\end{aligned}$$

This means

$$\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1 = \mathbb{N}(\mathfrak{a})\sqrt{d_k}.$$

At last, we use this result to produce the desired expression:

$$\begin{aligned}\mathrm{tr}(\omega_2)\omega_1 - \mathrm{tr}(\omega_1)\omega_2 &= (\omega_2 + \overline{\omega}_2)\omega_1 - (\omega_1 + \overline{\omega}_1)\omega_2 \\ &= \overline{\omega}_2\omega_1 - \overline{\omega}_1\omega_2 \\ &= \mathbb{N}(\mathfrak{a})\sqrt{d_k}.\end{aligned}$$

□

**Lemma 5.4.** *Let $C \in Cl_{\mathfrak{m}}(k)$ and let $\mathfrak{c}$ be an integral ideal in $C$. Recall that $\mathfrak{d} := \mathfrak{d}_{k/\mathbb{Q}}$ and let $\mathfrak{a} = \mathfrak{c}\mathfrak{m}^{-1}\mathfrak{d}^{-1}$. Choose $\omega_1/\omega_2 \in \mathbb{H}$ such that $\mathfrak{a} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Furthermore, let $\widetilde{f}$ be a homogeneous function of degree 0 on enhanced elliptic curves. Then we have*

*(1)* $\mathbb{C}/\mathfrak{a} \cong \mathbb{C}/\overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}$,

*(2)* $\widetilde{f}(1 + \overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}, \mathbb{C}/\overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}) = \widetilde{f}(tr(\omega_2)\omega_1 - tr(\omega_1)\omega_2 + \mathfrak{a}, \mathbb{C}/\mathfrak{a})$.

*Proof.* Define a map $\phi : \mathbb{C} \to \mathbb{C}$ via $\phi(z) = \mathbb{N}(\mathfrak{a})\sqrt{d_k}$. Recall from Theorem 4.1 that $\phi$ is an isomorphism between $\mathbb{C}/\overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}$ and $\mathbb{C}/\mathfrak{a}$ if $\mathfrak{a} = \mathbb{N}(\mathfrak{a})\sqrt{d_k} \cdot \overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}$. Noting that $\mathfrak{d} = \sqrt{d_k}\mathcal{O}_k$ and that $\mathbb{N}(\mathfrak{a}) = \mathfrak{a}\overline{\mathfrak{a}}$, we have

$$\mathbb{N}(\mathfrak{a})\sqrt{d_k} \cdot \overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1} = \mathfrak{a}\overline{\mathfrak{a}}\mathfrak{d}\overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1} = \mathfrak{a}. \tag{5.3}$$

Thus, the first statement of the lemma is proved. Now, using (5.3) along with Lemma 5.3 gives

$$\mathbb{N}(\mathfrak{a})\sqrt{d_k}(1 + \overline{\mathfrak{m}} \cdot \overline{\mathfrak{c}}^{-1}) = \mathrm{tr}(\omega_2)\omega_1 - \mathrm{tr}(\omega_1)\omega_2 + \mathfrak{a}. \tag{5.4}$$

Finally, (5.3) and (5.4) together, along with the fact that $\widetilde{f}$ is homogeneous of degree 0, proves the second statement of the theorem. □

**Theorem 5.5.** *Let $\psi$ be a proper character of the ray class group $Cl_{\mathfrak{m}}(k)$. Then*

$$L'_{\mathfrak{m}}(0,\psi) = -\frac{1}{12N_{\mathfrak{m}}w_{\mathfrak{m}}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(R)\log|g_{\mathfrak{m}}(R)|^2.$$

*where $N_{\mathfrak{m}}$ is the smallest positive integer in $\mathfrak{m}$, and $w_{\mathfrak{m}}$ denotes the number of roots of unity in $k$ which are congruent to 1 modulo $\mathfrak{m}$.*

*Proof.* This theorem follows from (3.31), and thus we begin by recalling the notation used there. Fix $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$ such that $(\mathfrak{m}, y\mathfrak{m}\mathfrak{d}) = 1$. For each class $R \in Cl_{\mathfrak{m}}(k)$ fix an integral ideal $\mathfrak{b}_R \in R$ and choose $\omega_1, \omega_2$ such that $\tau_R := \omega_1/\omega_2 \in \mathbb{H}$ and $y\mathfrak{b}_R = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Letting $\mathfrak{a} = y\mathfrak{m}\mathfrak{d}$, equation (3.31) gives

$$L'_{\mathfrak{m}}(0,\psi) = -\frac{1}{12N_{\mathfrak{m}}w_{\mathfrak{m}}} \sum_{R\in Cl_{\mathfrak{m}}(k)} \psi(\mathfrak{a}\mathfrak{b}_R)\log\left|g_{-v,u}(\tau_R)^{12N_{\mathfrak{m}}}\right|^2,$$

where $u = \mathrm{Tr}(\omega_1)$ and $v = \mathrm{Tr}(\omega_2)$. Now, let $\Lambda = y\mathfrak{b}_R$ and note that $\Lambda = \mathfrak{a}\mathfrak{b}_R\mathfrak{m}^{-1}\mathfrak{d}^{-1}$. Then, from the proof of Theorem 4.19, the value of the homogeneous function $\widetilde{g}^{12N_{\mathfrak{m}}}$ corresponding to $g_{(-v,u)}(\omega_1/\omega_2)^{12N_{\mathfrak{m}}}$ is given by

$$\widetilde{g}(-v\omega_1 + u\omega_2 + \Lambda, \mathbb{C}/\Lambda)^{12N_{\mathfrak{m}}}.$$

Note that for this construction to be well defined $-v\omega_1 + u\omega_2 + \Lambda$ must be a point of order $N_{\mathfrak{m}}$ on $\mathbb{C}/\Lambda$. This is immediate from Lemma 5.4, however, because letting $\mathfrak{c} = \mathfrak{a}\mathfrak{b}_R$ we have that $\mathbb{C}/\overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1} \cong \mathbb{C}/\Lambda$, and the isomorphism takes $1 + \overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1}$ to $-v\omega_1 + u\omega_2 + \Lambda$. Since the former is a point of order $N_{\mathfrak{m}}$, so must be the latter. Now, using Lemma 5.4 again we have

$$\begin{aligned} g_{-v,u}(\tau_R)^{12N_{\mathfrak{m}}} &= \widetilde{g}(v\omega_1 - u\omega_2 + \Lambda, \mathbb{C}/\Lambda)^{12N_{\mathfrak{m}}} \\ &= \widetilde{g}(1 + \overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1}, \mathbb{C}/\overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1})^{12N_{\overline{\mathfrak{m}}}} \\ &= g_{\overline{\mathfrak{m}}}(\overline{R'}), \end{aligned}$$

where $R'$ is the class of $\mathfrak{a}\mathfrak{b}_R$. Now, let $\mathfrak{c} = \mathfrak{a}\mathfrak{b}_R$. Using the definition of the Siegel-Ramachandra invariant from (5.2) we have,

$$g_{\overline{\mathfrak{m}}}(\overline{R'}) = \widetilde{g}(1 + \overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1}, \mathbb{C}/\overline{\mathfrak{m}}\cdot\overline{\mathfrak{c}}^{-1})^{12N_{\overline{\mathfrak{m}}}}.$$

Noting that $N_{\overline{\mathfrak{m}}} = N_{\mathfrak{m}}$ and using Theorem 4.21 produces

$$\begin{aligned} g_{\overline{\mathfrak{m}}}(\overline{R'}) &= \overline{\widetilde{g}(1+\mathfrak{m}\mathfrak{c}^{-1}, \mathbb{C}/\mathfrak{m}\mathfrak{c}^{-1})}^{12N_{\mathfrak{m}}} \\ &= \overline{g_{\mathfrak{m}}(R')} \end{aligned}$$

Since $|\overline{g_{\mathfrak{m}}(R')}| = |g_{\mathfrak{m}}(R')|$, this completes the proof. □

Recall that $G_{\mathfrak{m}}$ denotes the Galois group of the ray class field extension $k_{\mathfrak{m}}/k$. Using properties of the Siegel-Ramachandra invariant, we can translate the result of Theorem 5.5, which is valid for proper characters of $Cl_{\mathfrak{m}}(k)$, to an analogous result that is valid for all characters of $G_{\mathfrak{m}}$.

**Theorem 5.6.** *Let $S_{\mathfrak{m}} = \{\mathfrak{p} \mid \mathfrak{m}\} \cup \{\infty\}$ where $v_0 := \infty$ is the unique infinite place of $k$. Furthermore, let $g_{\mathfrak{m}} := g_{\mathfrak{m}}(1)$. Then*

$$L'_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(0,\chi) = -\frac{1}{12N_{\mathfrak{m}}w_{\mathfrak{m}}} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma) \log |g_{\mathfrak{m}}^{\sigma}|^2$$

*for all $\chi \in G_{\mathfrak{m}}$.*

*Proof.* Begin with a character $\chi : G_{\mathfrak{m}} \to \mathbb{C}^{\times}$. Let $k_{\mathfrak{m}}^{\ker\chi}$ be the subfield of $k_{\mathfrak{m}}$ fixed by $\ker\chi$, and let $\mathfrak{f}_{\chi}$ be the conductor of the extension $k_{\mathfrak{m}}^{\ker\chi}/k$. Then, $\chi$ gives rise to a proper character on $Cl_{\mathfrak{f}_{\chi}}(k)$ as follows. Consider the tower of fields $k \subseteq k_{\mathfrak{m}}^{\ker\chi} \subseteq k_{\mathfrak{m}}$. Let $\Gamma := \mathrm{Gal}(k_{\mathfrak{m}}^{\ker\chi}/k)$ and note that $\ker\chi = \mathrm{Gal}(k_{\mathfrak{m}}/k_{\mathfrak{m}}^{\ker\chi})$. Then, $\chi$ induces an injective character $\widetilde{\chi} : \Gamma \to \mathbb{C}^{\times}$ since $\Gamma \cong G_{\mathfrak{m}}/\ker\chi$, and we obtain a character $\psi$ on $Cl_{\mathfrak{f}_{\chi}}(k)$ defined by the following composition of maps

$$Cl_{\mathfrak{f}_{\chi}}(k) \xrightarrow{\pi} \Gamma \xhookrightarrow{\widetilde{\chi}} \mathbb{C}^{\times},$$

where $\pi$ is the composition of the natural projection of $G_{\mathfrak{f}_{\chi}}$ onto $\Gamma$ with the Artin isomorphism $\varphi$. By Theorem 1.16 the character $\psi$ is proper. Moreover, by Theorem 1.21 we have

$$L_{k_{\mathfrak{m}}/k}(s,\chi) = L_{\mathfrak{f}_{\chi}}(s,\psi).$$

Then Theorem 5.5 gives

$$\begin{aligned} L'_{k_{\mathfrak{m}}/k}(0,\chi) &= L'_{\mathfrak{f}_\chi}(0,\psi) \\ &= -\frac{1}{12N_{\mathfrak{f}_\chi}w_{\mathfrak{f}_\chi}} \sum_{C\in Cl_{\mathfrak{f}_\chi}(k)} \psi(C)\log|g_{\mathfrak{f}_\chi}(C)|^2 \\ &= -\frac{1}{12N_{\mathfrak{f}_\chi}w_{\mathfrak{f}_\chi}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \psi(\varphi^{-1}(\sigma))\log|g_{\mathfrak{f}_\chi}(\varphi^{-1}(\sigma))|^2 \\ &= -\frac{1}{12N_{\mathfrak{f}_\chi}w_{\mathfrak{f}_\chi}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\log|g^{\sigma}_{\mathfrak{f}_\chi}|^2, \end{aligned} \tag{5.5}$$

where the last step follows from (SR2) since

$$g_{\mathfrak{f}_\chi}(\varphi^{-1}(\sigma)) = g_{\mathfrak{f}_\chi}(1)^{\varphi(\varphi^{-1}(\sigma))} = g^{\sigma}_{\mathfrak{f}_\chi}.$$

Now, recall that $S_{\mathfrak{m}} = \{\mathfrak{p} \mid \mathfrak{m}\} \cup \{\infty\}$. Thus, we have

$$L_{k_{\mathfrak{m}}/k}(s,\chi) = L_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(s,\chi)\cdot \prod_{\mathfrak{p}|\mathfrak{m}} E_{\mathfrak{p}}(s,\chi)^{-1},$$

where $E_{\mathfrak{p}}(s,\chi)$ is the Euler factor for $\mathfrak{p}$ defined in (1.14). From the argument in Theorem 1.21 it follows that $\chi$ is unramified at a prime ideal $\mathfrak{p}$ if and only if $\mathfrak{p} \nmid \mathfrak{f}_\chi$. So, $E_{\mathfrak{p}}(s,\chi) = 1$ unless $\mathfrak{p} \nmid \mathfrak{f}_\chi$. This means

$$L_{k_{\mathfrak{m}}/k}(s,\chi) = L_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(s,\chi)\cdot \prod_{\substack{\mathfrak{p}|\mathfrak{m}\\ \mathfrak{p}\nmid\mathfrak{f}_\chi}} \left(1-\frac{\chi(\sigma_{\mathfrak{p}})}{\mathbb{N}(\mathfrak{p})^s}\right).$$

Note that all of the primes in the product on the right are unramified in the extension $k_{\mathfrak{f}_\chi}/k$. Thus, the argument used in Lemma 2.6 can be applied to (5.5) to determine the effect on $L'_{k_{\mathfrak{m}}/k}(0,\chi)$ of multiplying by the Euler factor for $\mathfrak{p}$. Using this along with (SR6) produces

$$\begin{aligned} L'_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(0,\chi) &= -\frac{1}{12N_{\mathfrak{f}_\chi} w_{\mathfrak{f}_\chi}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\log|(g_{\mathfrak{f}_\chi}^{\alpha})^{\sigma}|^2 \\ &= -\frac{1}{12N_{\mathfrak{m}} w_{\mathfrak{f}_\chi}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\left(\frac{N_{\mathfrak{m}}}{N_{\mathfrak{f}_\chi}}\log|(g_{\mathfrak{f}_\chi}^{\alpha})^{\sigma}|^2\right) \\ &= -\frac{1}{12N_{\mathfrak{m}} w_{\mathfrak{f}_\chi}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\log\left|\left(N_{k_{\mathfrak{m}}/k_{\mathfrak{f}_\chi}}(g_{\mathfrak{m}})^{w_{\mathfrak{f}_\chi}/w_{\mathfrak{m}}}\right)^{\sigma}\right|^2 \\ &= -\frac{1}{12N_{\mathfrak{m}} w_{\mathfrak{m}}} \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\log\left|\left(N_{k_{\mathfrak{m}}/k_{\mathfrak{f}_\chi}}(g_{\mathfrak{m}})\right)^{\sigma}\right|^2. \end{aligned}$$

The same argument which lead to (2.5) now implies that

$$\sum_{\tau\in G_{\mathfrak{m}}} \chi(\tau)\log|g_{\mathfrak{m}}^{\tau}|^2 = \sum_{\sigma\in G_{\mathfrak{f}_\chi}} \chi(\sigma)\log\left|\left(N_{k_{\mathfrak{m}}/k_{\mathfrak{f}_\chi}}(g_{\mathfrak{m}})\right)^{\sigma}\right|^2,$$

which concludes the proof. □

**Theorem 5.7.** *Conjecture 2.1 is true when $k$ is a quadratic imaginary number field.*

*Proof.* Let $\mathfrak{f}$ be the conductor of the abelian extension $K/k$ and let $\mathfrak{m} = \mathfrak{f}^a$ where $a$ is a positive integer large enough that $w_{\mathfrak{m}} = 1$. Such an $a$ is guaranteed to exist because if $\zeta \in \mu(k_{\mathfrak{m}})$ and $\zeta \equiv 1 \pmod{\mathfrak{m}}$ this means that $\operatorname{ord}_{\mathfrak{p}}(\zeta - 1) \geq \operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})$ for all prime ideals $\mathfrak{p} \mid \mathfrak{m}$. The right hand side can be made arbitrarily large by increasing $a$. However, since $\mu(k_{\mathfrak{m}})$ is finite, the left hand side is bounded except for $\zeta = 1$.

Theorem 1.14 implies that the prime ideals $\mathfrak{p}$ which ramify in $K/k$ are precisely those for which $\mathfrak{p} \mid \mathfrak{f}$. Since $\mathfrak{m}$ is just a power of $\mathfrak{f}$ it follows that these are also the primes which ramify in $k_{\mathfrak{m}}/k$. Thus, Lemmas 2.6 and 2.8 imply that it suffices to prove Stark's conjecture for the extension $k_{\mathfrak{m}}/k$ using the minimal set $S_{\mathfrak{m}} = \{\infty\} \cup \{\mathfrak{p} \mid \mathfrak{m}\}$.

We now verify (2.1) using the properties of the Siegel-Ramachandra invariant in Theorem 5.2, along with Theorem 5.6. From the latter, since $w_{\mathfrak{m}} = 1$, we have

$$L'_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(0,\chi) = -\frac{1}{12N_{\mathfrak{m}}} \sum_{\sigma\in G_{\mathfrak{m}}} \chi(\sigma)\log|g_{\mathfrak{m}}^{\sigma}|^2,$$

for all $\chi \in \widehat{G}_{\mathfrak{m}}$. From (SR1) and (SR5) it follows that the element $g_{\mathfrak{m}} \in k_{\mathfrak{m}}^{\times}$ and that $k_{\mathfrak{m}}(g_{\mathfrak{m}}^{1/12N_{\mathfrak{m}}})/k$ is abelian. Thus, using Lemma 2.7 with $\alpha = g_{\mathfrak{m}}$ and $m = 12N_{\mathfrak{m}}$, there exists $\varepsilon_{\mathfrak{m}} \in k_{\mathfrak{m}}^{\times}$ such that

$$g_{\mathfrak{m}}^{w_{k_{\mathfrak{m}}}} = \zeta \cdot \varepsilon_{\mathfrak{m}}^{12N_{\mathfrak{m}}} \tag{5.6}$$

for some $\zeta \in \mu(k_{\mathfrak{m}})$. So, from the expression for $L'_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(0,\chi)$ above, we have

$$\begin{aligned} L'_{k_{\mathfrak{m}}/k,S_{\mathfrak{m}}}(0,\chi) &= -\frac{w_{k_{\mathfrak{m}}}}{12N_{\mathfrak{m}}w_{k_{\mathfrak{m}}}} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma) \log |g_{\mathfrak{m}}^{\sigma}|^2 \\ &= -\frac{1}{12N_{\mathfrak{m}}w_{k_{\mathfrak{m}}}} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma) \log \left|(g_{\mathfrak{m}}^{w_{k_{\mathfrak{m}}}})^{\sigma}\right|^2 \\ &= -\frac{1}{w_{k_{\mathfrak{m}}}} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma) \log |\varepsilon_{\mathfrak{m}}^{\sigma}|^2 . \end{aligned}$$

Moreover, $g_{\mathfrak{m}}$ is an $S_{\mathfrak{m}}$-unit by (SR3), so (5.6) implies that $\varepsilon_{\mathfrak{m}}$ is also an $S_{\mathfrak{m}}$-unit. In addition,

$$|\varepsilon_{\mathfrak{m}}^{\sigma}|^2 = |\varepsilon_{\mathfrak{m}}^{\sigma}|_{w_0},$$

for some place $w_0$ of $k_{\mathfrak{m}}$ lying above $v_0 = \infty$. This is enough to verify (2.1), since Lemma 2.4 shows that the truth of $St(k_{\mathfrak{m}}/k, S_{\mathfrak{m}}, v_0, w_0)$ is independent of the choice of $w_0$ lying above $v_0$.

It remains to show that $\varepsilon_{\mathfrak{m}}$ satisfies the absolute value conditions of Stark's conjecture. If $|S_{\mathfrak{m}}| \geq 3$ then this means $\mathfrak{m}$ has at least two distinct prime divisors, so $g_{\mathfrak{m}}$ is a unit by (SR4). Therefore, $\varepsilon_{\mathfrak{m}}$ is as well, meaning $|\varepsilon_{\mathfrak{m}}|_w = 1$ for all places $w$ of $k_{\mathfrak{m}}$ not lying above $\infty$. If $|S_{\mathfrak{m}}| = 2$, this means $\mathfrak{m}$ is the power of a single prime ideal $\mathfrak{p}$. Thus, (SR4) implies $g_{\mathfrak{m}}/g_{\mathfrak{m}}^{\sigma}$ is a unit for all $\sigma \in G_{\mathfrak{m}}$. It follows that $\varepsilon_{\mathfrak{m}}/\varepsilon_{\mathfrak{m}}^{\sigma}$ is also a unit for all $\sigma \in G_{\mathfrak{m}}$, meaning

$$|\varepsilon_{\mathfrak{m}}^{\sigma^{-1}}|_w = |\varepsilon_{\mathfrak{m}}|_{w^{\sigma}} = |\varepsilon_{\mathfrak{m}}|_w$$

for all places $w$ of $k_{\mathfrak{m}}$ not lying above $\infty$ and for all $\sigma \in G_{\mathfrak{m}}$. This verifies the last of the absolute value conditions and, thus, proves that $\varepsilon_{\mathfrak{m}}$ is a Stark unit. $\square$

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